

REALIZATIONS OF GLOBALLY EXCEPTIONAL $\mathbb{Z}_2 \times \mathbb{Z}_2$ - SYMMETRIC SPACES

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Dedicated to Professor Ichiro Yokota on the occasion of his eighty-eighth birthday

ABSTRACT. In [3], a classification is given of the exceptional $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces G/K , where G is an exceptional compact Lie group or $Spin(8)$, and moreover the structure of K is determined as Lie algebra. In the present article, we give a pair of commuting involutive automorphisms (involutions) $\tilde{\sigma}, \tilde{\tau}$ of G concretely and determine the structure of group $G^\sigma \cap G^\tau$ corresponding to Lie algebra $\mathfrak{g}^\sigma \cap \mathfrak{g}^\tau$, where G is an exceptional compact Lie group. Thereby, we realize exceptional $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces, globally.

1. INTRODUCTION

According to the article [3], the notion of Γ -symmetric spaces introduced by Lutz [4], is a generalization of the classical notion of a symmetric space, where Γ is a finite abelian group. (As for the definition of Γ -symmetric space, see [1].) In the case $\Gamma = \mathbb{Z}_2$ this is the classical definition of symmetric spaces, and in the case $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ we say that this is $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space. Now, the definition of $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space in [3] is as follows.

Definition. A homogeneous space G/K is $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space if there are $\tilde{\sigma}, \tilde{\tau} \in \text{Aut}(G) \setminus \{\text{id}_G\}$ such that $\tilde{\sigma}^2 = \tilde{\tau}^2 = \text{id}_G$, $\tilde{\sigma} \neq \tilde{\tau}$ and $\tilde{\sigma}\tilde{\tau} = \tilde{\tau}\tilde{\sigma}$ such that $(G^\sigma \cap G^\tau)_0 \subseteq K \subseteq G^\sigma \cap G^\tau$, where G^σ (resp. G^τ) is a fixed points subgroup of G by $\tilde{\sigma}$ (resp. $\tilde{\tau}$) and $(G^\sigma \cap G^\tau)_0$ is a connected component containing 1 of $G^\sigma \cap G^\tau$. (Hereafter id_G is abbreviated as 1.)

The main purpose of this article is to give a pair of different involutive automorphisms $\tilde{\sigma}, \tilde{\tau}$ in G and to determine the structure of the group $G^\sigma \cap G^\tau$ corresponding to $\mathfrak{k} = \mathfrak{g}^\sigma \cap \mathfrak{g}^\tau$ in the second column of Table 1, where G is a simply connected compact exceptional Lie group G_2, F_4, E_6, E_7 or E_8 . Thereby, we realize exceptional $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces, globally. We call those spaces "Globally exceptional $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces". Moreover we confirm all types $(G/G^\sigma, G/G^\tau, G/G^{\sigma\tau})$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces determined by Andreas Kollross, globally. For example, since it follows from the triple group isomorphisms $(E_6)^{\lambda\gamma} \cong Sp(4)/\mathbb{Z}_2$, $(E_6)^{\lambda\gamma\sigma} \cong (E_6)^{\lambda\gamma} \cong Sp(4)/\mathbb{Z}_2$, $(E_6)^{(\lambda\gamma)(\lambda\gamma\sigma)} = (E_6)^\sigma \cong (U(1) \times Spin(10))/\mathbb{Z}_4$ that $E_6/(E_6)^{\lambda\gamma}$, $E_6/(E_6)^{\lambda\gamma\sigma}$, $E_6/(E_6)^{(\lambda\gamma)(\lambda\gamma\sigma)}$ are the symmetric spaces of type EI, EI, EIII, respectively. Then the globally $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space of this type is called type EI-EI-EIII, and denote EI-EI-EIII by abbreviated form EI-I-III. In addition, when σ and τ are conjugate in G , we give explicitly the element $\delta \in G$ such that $\sigma = \delta\tau\delta^{-1}$ except for three cases in E_8 .

This article is closely in connection with the preceding articles [5], [6], [7], [8], [10], [11], [12] and [13], and may be a continuation of those in some sense.

J.-S.H and J.U [2] classified the Klein four subgroups Γ of $\text{Aut}(u_0)$ for each compact Lie algebra u_0 by calculating the symmetric subgroups $\text{Aut}(u_0)^\theta$ ($\theta \in \text{Aut}(u_0)$ is an involutive automorphism) and their involution classes, and determined the fixed point subgroup $\text{Aut}(u_0)^\Gamma$. In general, suppose a group G is simply connected, we have $\text{Aut}(G) \cong \text{Aut}(\mathfrak{g})$

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([9]. \mathfrak{g} is the Lie algebra of G), moreover when the center $z(G)$ of G is trivial, it is well known that $G \subset \text{Aut}(G)$. Since the exceptional compact Lie groups $G = G_2, F_4, E_8$ are simply connected and these $z(G)$ are trivial, we see that $G \subset \text{Aut}(G) \cong \text{Aut}(\mathfrak{g})$. Hence, for $G = G_2, F_4, E_8$, our results of $G^\sigma \cap G^\tau$ in Table 1 are realized as the subgroups of the results of fixed point subgroups of Klein four subgroups in exceptional case of [2].

In [2], they had approached the ends by using root system of \mathfrak{u}_0 . On the other hand, we define the mappings between groups explicitly, and give the proofs of isomorphism of group by using homomorphism theorem as elementary approach. The author would like to say that this is one of features about this article.

For $G = G_2, F_4, E_6, E_7$, and E_8 , our results are as follows.

Type	\mathfrak{g}	$\mathfrak{k}(\cong \mathfrak{g}^\sigma \cap \mathfrak{g}^\tau)$	Involutions	$G^\sigma \cap G^\tau$
G-G-G	\mathfrak{g}_2	$i\mathbf{R} \oplus i\mathbf{R}$	γ, γ_H	$(U(1) \times U(1))/\mathbf{Z}_2 \rtimes \{1, \gamma_C\}$
FI-I-I	\mathfrak{f}_4	$\mathfrak{u}(3) \oplus i\mathbf{R}$	γ, γ_H	$(U(1) \times U(1) \times SU(3))/\mathbf{Z}_3 \rtimes \{1, \gamma_C\}$
FI-I-II	\mathfrak{f}_4	$\mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$	$\gamma, \gamma\sigma$	$(Sp(1) \times Sp(1) \times Sp(2))/\mathbf{Z}_2$
FII-II-II	\mathfrak{f}_4	$\mathfrak{so}(8)$	σ, σ'	$Spin(8)$
EI-I-II	\mathfrak{e}_6	$\mathfrak{so}(6) \oplus i\mathbf{R}$	$\lambda\gamma, \lambda\gamma\gamma_C$	$(U(1) \times SO(6))/\mathbf{Z}_2 \rtimes \{1, \gamma_H\}$
EI-I-III	\mathfrak{e}_6	$\mathfrak{sp}(2) \oplus \mathfrak{sp}(2)$	$\lambda\gamma, \lambda\gamma\sigma$	$(Sp(2) \times Sp(2))/\mathbf{Z}_2 \rtimes \{1, \rho\}$
EI-II-IV	\mathfrak{e}_6	$\mathfrak{sp}(3) \oplus \mathfrak{sp}(1)$	$\lambda\gamma, \gamma$	$(Sp(1) \times Sp(3))/\mathbf{Z}_2$
EII-II-II	\mathfrak{e}_6	$\mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus i\mathbf{R} \oplus i\mathbf{R}$	γ, γ_H	$(U(1) \times U(1) \times SU(3) \times SU(3))/\mathbf{Z}_3 \rtimes \{1, \gamma_C\}$
EII-II-III	\mathfrak{e}_6	$\mathfrak{su}(4) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus i\mathbf{R}$	$\gamma, \sigma\gamma$	$(Sp(1) \times Sp(1) \times U(1) \times SU(4))/(\mathbf{Z}_2 \times \mathbf{Z}_4)$
EII-III-III	\mathfrak{e}_6	$\mathfrak{su}(5) \oplus i\mathbf{R} \oplus i\mathbf{R}$	$\gamma, \gamma_H\rho_2$	$(U(1) \times U(1) \times SU(5))/(\mathbf{Z}_2 \times \mathbf{Z}_5)$
EIII-III-III	\mathfrak{e}_6	$\mathfrak{so}(8) \oplus i\mathbf{R} \oplus i\mathbf{R}$	σ, σ'	$(U(1) \times U(1) \times Spin(8))/(\mathbf{Z}_2 \times \mathbf{Z}_4)$
EIII-IV-IV	\mathfrak{e}_6	$\mathfrak{so}(9)$	λ, σ	$Spin(9)$
EV-V-V	\mathfrak{e}_7	$\mathfrak{so}(8)$	$\lambda\gamma, \iota\gamma_C$	$SO(8)/\mathbf{Z}_2 \times \{1, -1\}$
EV-V-VI	\mathfrak{e}_7	$\mathfrak{su}(4) \oplus \mathfrak{su}(4) \oplus i\mathbf{R}$	$\lambda\gamma, \lambda\gamma\sigma$	$(U(1) \times SU(4) \times SU(4))/(\mathbf{Z}_2 \times \mathbf{Z}_4) \rtimes \{1, \varepsilon\}$
EV-V-VII	\mathfrak{e}_7	$\mathfrak{sp}(4)$	$\lambda\gamma, \iota\lambda\gamma$	$Sp(4)/\mathbf{Z}_2 \times \{1, -1\}$
EV-VI-VII	\mathfrak{e}_7	$\mathfrak{su}(6) \oplus \mathfrak{sp}(1) \oplus i\mathbf{R}$	$\lambda\gamma, \gamma$	$(U(1) \times SU(2) \times SU(6))/\mathbf{Z}_{24}$
EVI-VI-VI	\mathfrak{e}_7 \mathfrak{e}_7	$\mathfrak{so}(8) \oplus \mathfrak{so}(4) \oplus \mathfrak{sp}(1)$ $\mathfrak{u}(1) \oplus \mathfrak{su}(6) \oplus i\mathbf{R}$	$\gamma, -\sigma$ γ, γ_H	$(SU(2) \times Spin(4) \times Spin(8))/(\mathbf{Z}_2 \times \mathbf{Z}_2)$ $(U(1) \times U(1) \times SU(6))/\mathbf{Z}_3 \rtimes \{1, \gamma_C\}$
EVI-VII-VII	\mathfrak{e}_7	$\mathfrak{so}(10) \oplus i\mathbf{R} \oplus i\mathbf{R}$	$-\sigma, \iota$	$(U(1) \times U(1) \times Spin(10))/\mathbf{Z}_{12}$
EVII-VII-VII	\mathfrak{e}_7	\mathfrak{f}_4	ι, λ	$F_4 \times \{1, -1\}$
EVIII-VIII-VIII	\mathfrak{e}_8	$\mathfrak{so}(8) \oplus \mathfrak{so}(8)$	σ, σ'	$(Spin(8) \times Spin(8))/(\mathbf{Z}_2 \times \mathbf{Z}_2)$
EVIII-VIII-IX	\mathfrak{e}_8	$\mathfrak{su}(8) \oplus i\mathbf{R}$	$\lambda_\omega\gamma, \lambda_\omega\gamma v$	$(SO(2) \times SU(8))/\mathbf{Z}_4 \rtimes \{1, \rho_v\}$
EVIII-IX-IX	\mathfrak{e}_8	$\mathfrak{so}(12) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$	σ, v	$(SU(2) \times SU(2) \times Spin(12))/\mathbf{Z}_4$
EIX-IX-IX	\mathfrak{e}_8	$\mathfrak{e}_6 \oplus i\mathbf{R} \oplus i\mathbf{R}$	v, ι_ω	$(SO(2) \times U(1) \times E_6)/\mathbf{Z}_6 \rtimes \{1, v\}$

Table 1. Globally exceptional $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces

Remark. In the forth column, we omit a sign \sim , for example $\tilde{\gamma}$ is denoted by γ . In the fifth column, a sign \rtimes means semi-direct product of groups, for example $(U(1) \times SO(6))/\mathbb{Z}_2 \rtimes \{1, \gamma_H\}$.

2. PRELIMINARIES

We give the definitions of the simply connected compact exceptional Lie groups used in this article, and we state general notes for notation.

2.1. Cayley algebra and compact Lie group of type G_2 . Let $\mathbb{C} = \{e_0 = 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}_{\mathbb{R}}$ be the division Cayley algebra. In \mathbb{C} , since the multiplication and the inner product are well known, these are omitted.

The simply connected compact Lie group of type G_2 is given by

$$G_2 = \{\alpha \in \text{Iso}_{\mathbb{R}}(\mathbb{C}) \mid \alpha(xy) = (\alpha x)(\alpha y)\}.$$

2.2. Exceptional Jordan algebra and compact Lie group of type F_4 . Let $\mathfrak{J}(3, \mathbb{C}) = \{X \in M(3, \mathbb{C}) \mid X^* = X\}$ be the exceptional Jordan algebra. In $\mathfrak{J}(3, \mathbb{C})$, the Jordan multiplication $X \circ Y$, the inner product (X, Y) and a cross multiplication $X \times Y$, called the Freudenthal multiplication, are defined by

$$\begin{aligned} X \circ Y &= \frac{1}{2}(XY + YX), \quad (X, Y) = \text{tr}(X \circ Y), \\ X \times Y &= \frac{1}{2}(2X \circ Y - \text{tr}(X)Y - \text{tr}(Y)X + (\text{tr}(X)\text{tr}(Y) - (X, Y))E), \end{aligned}$$

respectively, where E is the 3×3 unit matrix. Moreover, we define the trilinear form (X, Y, Z) , the determinant $\det X$ by

$$(X, Y, Z) = (X, Y \times Z), \quad \det X = \frac{1}{3}(X, X, X),$$

respectively, and briefly denote $\mathfrak{J}(3, \mathbb{C})$ by \mathfrak{J} .

The simply connected compact Lie group of type F_4 is given by

$$\begin{aligned} F_4 &= \{\alpha \in \text{Iso}_{\mathbb{R}}(\mathfrak{J}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\} \\ &= \{\alpha \in \text{Iso}_{\mathbb{R}}(\mathfrak{J}) \mid \alpha(X \times Y) = \alpha X \times \alpha Y\}. \end{aligned}$$

Then we have naturally the inclusion $G_2 \subset F_4$.

2.3. Complex exceptional Jordan algebra and Compact Lie group of type E_6 . Let $\mathfrak{J}(3, \mathbb{C})^{\mathbb{C}} = \{X \in M(3, \mathbb{C})^{\mathbb{C}} \mid X^* = X\}$ be the complexification of the exceptional Jordan algebra \mathfrak{J} . In $\mathfrak{J}(3, \mathbb{C})^{\mathbb{C}}$, as in \mathfrak{J} , we can also define the multiplication $X \circ Y$, $X \times Y$, the inner product (X, Y) , the trilinear forms (X, Y, Z) and the determinant $\det X$ in the same manner, and those have the same properties. The $\mathfrak{J}(3, \mathbb{C})^{\mathbb{C}}$ is called the complex exceptional Jordan algebra, and briefly denote $\mathfrak{J}(3, \mathbb{C})^{\mathbb{C}}$ by $\mathfrak{J}^{\mathbb{C}}$.

The simply connected compact Lie group of type E_6 is given by

$$E_6 = \{\alpha \in \text{Iso}_{\mathbb{C}}(\mathfrak{J}^{\mathbb{C}}) \mid \det \alpha X = \det X, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\},$$

where the Hermite inner product $\langle X, Y \rangle$ is defined by $(\tau X, Y)$ (τ is a complex conjugation in $\mathfrak{J}^{\mathbb{C}}$: $\tau(X + iY) = X - iY$, $X, Y \in \mathfrak{J}$).

Then we have naturally the inclusion $G_2 \subset F_4 \subset E_6$.

2.4. C -vector space and compact Lie group of type E_7 . We define a C -vector space \mathfrak{P}^C , called the Freudenthal C -vector space, by

$$\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C \oplus C$$

with the Hermite inner product

$$\langle P, Q \rangle = \langle X, Z \rangle + \langle Y, W \rangle + (\tau\xi)\zeta + (\tau\eta)\omega$$

for $P = (X, Y, \xi, \eta)$, $Q = (Z, W, \zeta, \omega) \in \mathfrak{P}^C$. For $\phi \in \mathfrak{e}_6^C$, $A, B \in \mathfrak{J}^C$ and $\nu \in C$, we define a C -linear mapping $\Phi(\phi, A, B, \nu) : \mathfrak{P}^C \rightarrow \mathfrak{P}^C$ by

$$\Phi(\phi, A, B, \nu) \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \phi X - \frac{1}{3}\nu X + 2B \times Y + \eta A \\ 2A \times X - {}^t\phi Y + \frac{1}{3}\nu Y + \xi B \\ (A, Y) + \nu\xi \\ (B, X) - \nu\eta \end{pmatrix},$$

where ${}^t\phi \in \mathfrak{e}_6^C$ is the transpose of ϕ with respect to the inner product (X, Y) : $({}^t\phi X, Y) = (X, \phi Y)$ (\mathfrak{e}_6^C is the complex Lie algebra of type E_6).

Moreover, for $P = (X, Y, \xi, \eta)$, $Q = (Z, W, \zeta, \omega) \in \mathfrak{P}^C$, we define a C -linear mapping $P \times Q : \mathfrak{P}^C \rightarrow \mathfrak{P}^C$ by

$$P \times Q = \Phi(\phi, A, B, \nu), \quad \begin{cases} \phi = -\frac{1}{2}(X \vee W + Z \vee Y) \\ A = -\frac{1}{4}(2Y \times W - \xi Z - \zeta X) \\ B = \frac{1}{4}(2X \times Z - \eta W - \omega Y) \\ \nu = \frac{1}{8}((X, W) + (Z, Y) - 3(\xi\omega + \zeta\eta)), \end{cases}$$

where $X \vee W \in \mathfrak{e}_6^C$ is defined by $(X \vee W)U = \frac{1}{2}(W, U)X + \frac{1}{6}(X, W)U - 2W \times (X \times U)$ for $U \in \mathfrak{J}^C$.

The simply connected compact Lie group of type E_7 is given by

$$E_7 = \{ \alpha \in \text{Isoc}(\mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle \}.$$

Then we have naturally the inclusion $G_2 \subset F_4 \subset E_6 \subset E_7$.

2.5. C -vector space and compact Lie group of type E_8 . We define a C -vector space \mathfrak{e}_8^C by

$$\mathfrak{e}_8^C = \mathfrak{e}_7^C \oplus \mathfrak{P}^C \oplus \mathfrak{P}^C \oplus C \oplus C \oplus C,$$

with the Lie bracket $[R_1, R_2]$, $R_k = (\Phi_k, P_k, Q_k, r_k, s_k, t_k)$, $k = 1, 2$, defined by

$$[(\Phi_1, P_1, Q_1, r_1, s_1, t_1), (\Phi_2, P_2, Q_2, r_2, s_2, t_2)] = (\Phi, P, Q, r, s, t),$$

$$\begin{cases} \Phi = [\Phi_1, \Phi_2] + P_1 \times Q_2 - P_2 \times Q_1 \\ Q = \Phi_1 P_2 - \Phi_2 P_1 + r_1 P_2 - r_2 P_1 + s_1 Q_2 - s_2 Q_1 \\ P = \Phi_1 Q_2 - \Phi_2 Q_1 - r_1 Q_2 + r_2 Q_1 + t_1 P_2 - t_2 P_1 \\ r = -\frac{1}{8}\{P_1, Q_2\} + \frac{1}{8}\{P_2, Q_1\} + s_1 t_2 - s_2 t_1 \\ s = \frac{1}{4}\{P_1, P_2\} + 2r_1 s_2 - 2r_2 s_1 \\ t = -\frac{1}{4}\{Q_1, Q_2\} - 2r_1 t_2 + 2r_2 t_1, \end{cases}$$

where \mathfrak{e}_7^C is the complex Lie algebra of type E_7 , $\{P, Q\} = (X, W) - (Y, Z) + \xi\omega - \eta\zeta$, $P = (X, Y, \xi, \eta)$, $Q = (Z, W, \zeta, \omega) \in \mathfrak{P}^C$. Then \mathfrak{e}_8^C becomes the complex simple Lie algebra of

type E_8 .

Here, we define a C -linear transformation λ_ω of \mathfrak{e}_8^C by

$$\lambda_\omega(\Phi, P, Q, r, s, t) = (\lambda\Phi\lambda^{-1}, \lambda Q, -\lambda P, -r, -t, -s),$$

where λ of the right hand side is the C -linear transformation of \mathfrak{P}^C and is defined in Section 3.4.

Moreover, the complex conjugation in \mathfrak{e}_8^C is denoted by τ :

$$\tau(\Phi, P, Q, r, s, t) = (\tau\Phi\tau, \tau P, \tau Q, \tau r, \tau s, \tau t),$$

where τ in the right hand side is the usual complex conjugation in the complexification. Then we define a Hermitian inner product $\langle R_1, R_2 \rangle$ in \mathfrak{e}_8^C by

$$\langle R_1, R_2 \rangle = -\frac{1}{15}B_8(\tau\lambda_\omega R_1, R_2),$$

where B_8 is the Killing form of \mathfrak{e}_8^C (as for B_8 , see [10, Section E_8] in detail).

The simply connected compact Lie group E_8 are given by

$$E_8 = \{\alpha \in \text{Iso}_C(\mathfrak{e}_8^C) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2], \langle \alpha R_1, \alpha R_2 \rangle = \langle R_1, R_2 \rangle\}.$$

Then we have naturally the inclusion $G_2 \subset F_4 \subset E_6 \subset E_7 \subset E_8$.

Now, we state general notes of this article for notation. Let G be a group. For $\delta \in G$, $\tilde{\delta}$ denotes the inner automorphism induced by δ : $\tilde{\delta}(g) = \delta g \delta^{-1}$, $g \in G$, then $G^{\tilde{\delta}} = \{g \in G \mid \tilde{\delta}(g) = g\}$. Hereafter $G^{\tilde{\delta}}$ will be also written by G^δ . For $\alpha, \beta \in G$, when α and β are conjugate in G , we denote it by $\alpha \sim \beta$. Besides, we almost use the same notations as [10].

3. GLOBALLY EXCEPTIONAL SYMMETRIC SPACES OF TYPE I

In Table 2 below, the list of left half is classification of exceptional symmetric spaces that was found by Élie Cartan, on the other hand the list of right half is the results of group realizations corresponding to those. The structures of the groups G^ϱ below are well-known fact, however the explicit forms of involutive inner automorphisms ϱ are seldom known fact, so we write all in the following Table 2. The definitions of ϱ are written in the each section of this chapter. We remark that as in Table 1 we omit a sign \sim in the fifth column.

Type	\mathfrak{g}	$\mathfrak{k}(\cong \mathfrak{g}^\varrho)$	G	Involution ϱ	$K = G^\varrho$
G	\mathfrak{g}_2	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$	G_2	γ	$(Sp(1) \times Sp(1))/\mathbb{Z}_2$
FI	\mathfrak{f}_4	$\mathfrak{sp}(3) \oplus \mathfrak{sp}(1)$	F_4	γ	$(Sp(1) \times Sp(3))/\mathbb{Z}_2$
FII	\mathfrak{f}_4	$\mathfrak{so}(9)$	F_4	σ	$Spin(9)$
EI	\mathfrak{e}_6	$\mathfrak{sp}(4)$	E_6	$\lambda\gamma$	$Sp(4)/\mathbb{Z}_2$
EII	\mathfrak{e}_6	$\mathfrak{su}(6) \oplus \mathfrak{sp}(1)$	E_6	γ	$(Sp(1) \times SU(6))/\mathbb{Z}_2$
EIII	\mathfrak{e}_6	$\mathfrak{so}(10) \oplus i\mathbf{R}$	E_6	σ	$(U(1) \times Spin(10))/\mathbb{Z}_4$
EIV	\mathfrak{e}_6	\mathfrak{f}_4	E_6	λ	F_4

Type	\mathfrak{g}	$\mathfrak{k}(\cong \mathfrak{g}^\theta)$	G	Involution ϱ	$K = G^\varrho$
EV	\mathfrak{e}_7	$\mathfrak{su}(8)$	E_7	$\lambda\gamma$	$SU(8)/\mathbf{Z}_2$
EVI	\mathfrak{e}_7	$\mathfrak{so}(12) \oplus \mathfrak{sp}(1)$	E_7	γ	$(Sp(1) \times Spin(12))/\mathbf{Z}_2$
EVII	\mathfrak{e}_7	$\mathfrak{e}_6 \oplus i\mathbf{R}$	E_7	ι	$(U(1) \times E_6)/\mathbf{Z}_3$
EVIII	\mathfrak{e}_8	$\mathfrak{so}(16)$	E_8	$\lambda_\omega\gamma$	$Ss(16)$
EIX	\mathfrak{e}_8	$\mathfrak{e}_7 \oplus \mathfrak{sp}(1)$	E_8	ν	$(Sp(1) \times E_7)/\mathbf{Z}_2$

Table 2. Globally exceptional symmetric spaces of type I

In this chapter, each proof of the theorem is based on [10], and so whereas we omit the detail, refer to the each section in [10], which is written in every proof, for details.

3.1. Type G. Let $\mathfrak{C} = \mathbf{H} \oplus \mathbf{H}e_4$ be Cayley division algebra, where \mathbf{H} is the field of quaternion number and e_4 is one of basis in \mathfrak{C} .

We define an \mathbf{R} -linear transformation γ of \mathfrak{C} by

$$\gamma(a + be_4) = a - be_4, \quad a + be_4 \in \mathbf{H} \oplus \mathbf{H}e_4 = \mathfrak{C}.$$

Then we have $\gamma \in G_2$, $\gamma^2 = 1$. Hence γ induces involutive inner automorphism $\tilde{\gamma}$ of G_2 : $\tilde{\gamma}(\alpha) = \gamma\alpha\gamma, \alpha \in G_2$.

Now, the structure of the group $(G_2)^\gamma$ is as follows.

Theorem 3.1.1. [G] *The group $(G_2)^\gamma$ is isomorphism to the group $(Sp(1) \times Sp(1))/\mathbf{Z}_2$: $(G_2)^\gamma \cong (Sp(1) \times Sp(1))/\mathbf{Z}_2$, $\mathbf{Z}_2 = \{(1, 1), (-1, -1)\}$.*

Proof. We define a mapping $\varphi_G : Sp(1) \times Sp(1) \rightarrow (G_2)^\gamma$ by

$$\varphi_G(p, q)(m + ae_4) = qm\bar{q} + (pa\bar{q})e_4, \quad m + ae_4 \in \mathbf{H} \oplus \mathbf{H}e_4 = \mathfrak{C}.$$

This mapping induces the required isomorphism (see [10, Section 1.10]). \square

3.2. Types FI and FII. Let \mathfrak{J} be the exceptional Jordan algebra. An element $X \in \mathfrak{J}$ has the form

$$X = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \quad \xi_k \in \mathbf{R}, \quad x_k \in \mathfrak{C}, \quad k = 1, 2, 3.$$

Hereafter, in \mathfrak{J} , we use the following nations:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$F_1(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \bar{x} & 0 \end{pmatrix}, \quad F_2(x) = \begin{pmatrix} 0 & 0 & \bar{x} \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}, \quad F_3(x) = \begin{pmatrix} 0 & x & 0 \\ \bar{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We correspond such $X \in \mathfrak{J}$ to an element $M + \mathbf{a} \in \mathfrak{J}(3, \mathbf{H}) \oplus \mathbf{H}^3$ such that

$$\begin{pmatrix} \xi_1 & m_3 & \bar{m}_2 \\ \bar{m}_3 & \xi_2 & m_1 \\ m_2 & \bar{m}_1 & \xi_3 \end{pmatrix} + (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3),$$

where $x_k = m_k + a_k e_4 \in \mathbf{H} \oplus \mathbf{H}e_4 = \mathfrak{C}, k = 1, 2, 3$. Then $\mathfrak{J}(3, \mathbf{H}) \oplus \mathbf{H}^3$ has the Freudenthal multiplication and the inner product

$$(M + \mathbf{a}) \times (N + \mathbf{b}) = \left(M \times N - \frac{1}{2}(\mathbf{a}^* \mathbf{b} + \mathbf{b}^* \mathbf{a}) \right) - \frac{1}{2}(\mathbf{a}N + \mathbf{b}M),$$

$$(M + \mathbf{a}, N + \mathbf{b}) = (M, N) + 2(\mathbf{a}, \mathbf{b}),$$

where $(\mathbf{a}, \mathbf{b}) = \frac{1}{2}(\mathbf{a}\mathbf{b}^* + \mathbf{b}\mathbf{a}^*)$, corresponding those of \mathfrak{J} , that is, \mathfrak{J} is isomorphic to $\mathfrak{J}(3, \mathbf{H}) \oplus \mathbf{H}^3$ as algebra. From now on, we identify \mathfrak{J} with $\mathfrak{J}(3, \mathbf{H}) \oplus \mathbf{H}^3$.

We define \mathbf{R} -linear transformations γ, σ of \mathfrak{J} by

$$\gamma X = \begin{pmatrix} \xi_1 & \gamma x_3 & \overline{\gamma x_2} \\ \overline{\gamma x_3} & \xi_2 & \gamma x_1 \\ \gamma x_2 & \overline{\gamma x_1} & \xi_3 \end{pmatrix}, \quad \sigma X = \begin{pmatrix} \xi_1 & -x_3 & -\overline{x_2} \\ -\overline{x_3} & \xi_2 & x_1 \\ -x_2 & \overline{x_1} & \xi_3 \end{pmatrix}, \quad X \in \mathfrak{J},$$

respectively, where γ of right hand side is the same one as $\gamma \in G_2$. Then we have that $\gamma, \sigma \in F_4, \gamma^2 = \sigma^2 = 1$. Hence γ, σ induce involutive inner automorphisms $\tilde{\gamma}, \tilde{\sigma}$ of F_4 : $\tilde{\gamma}(\alpha) = \gamma\alpha\gamma, \tilde{\sigma}(\alpha) = \sigma\alpha\sigma, \alpha \in F_4$.

Now, the structures of groups $(F_4)^\gamma$ and $(F_4)^\sigma$ are as follows.

Theorem 3.2.1. [FI] *The group $(F_4)^\gamma$ is isomorphic to the group $(Sp(1) \times Sp(3))/\mathbf{Z}_2$: $(F_4)^\gamma \cong (Sp(1) \times Sp(3))/\mathbf{Z}_2, \mathbf{Z}_2 = \{(1, E), (-1, -E)\}$.*

Proof. We define a mapping $\varphi_{\text{FI}} : Sp(1) \times Sp(3) \rightarrow (F_4)^\gamma$ by

$$\varphi_{\text{FI}}(p, A)(M + \mathbf{a}) = AMA^* + p\mathbf{a}A^*, \quad M + \mathbf{a} \in \mathfrak{J}(3, \mathbf{H}) \oplus \mathbf{H}^3 = \mathfrak{J}.$$

This mapping induces the required isomorphism ([10, Section 2.11]). \square

Theorem 3.2.2. [FII] *The group $(F_4)^\sigma$ is isomorphic to the group $Spin(9): (F_4)^\sigma \cong Spin(9)$.*

Proof. From [10, Theorem 2.7.4], we have $(F_4)_{E_1} \cong Spin(9)$, so by proving that $(F_4)^\sigma \cong (F_4)_{E_1}$ ([10, Theorem 2.9.1]) we have the required isomorphism (see [10, Sections 2.7, 2.9] in detail). \square

3.3. Types EI, EII, EIII and EIV. Let \mathfrak{J}^C be the complex exceptional Jordan algebra. The complex conjugation τ of \mathfrak{J}^C satisfies the equalities:

$$\tau(X \circ Y) = \tau X \circ \tau Y, \quad \tau(X \times Y) = \tau X \times \tau Y, \quad X, Y \in \mathfrak{J}^C.$$

Here, we define an involutive automorphism λ of E_6 by

$$\lambda(\alpha) = {}^t\alpha^{-1}, \quad \alpha \in E_6.$$

Then, from the definition of transpose: $({}^t\alpha X, Y) = (X, \alpha Y)$, we see that

$$\lambda(\alpha) = \tau\alpha\tau, \quad \alpha \in E_6.$$

Let the C -linear transformations γ, σ of \mathfrak{J}^C be the complexification of $\gamma \in G_2 \subset F_4, \sigma \in F_4$. Then we have that $\gamma, \sigma \in E_6, \gamma^2 = \sigma^2 = 1$. Hence, as in F_4 , the group E_6 has involutive inner automorphisms $\tilde{\gamma}, \tilde{\sigma}$ induced by γ, σ : $\tilde{\gamma}(\alpha) = \gamma\alpha\gamma, \tilde{\sigma}(\alpha) = \sigma\alpha\sigma, \alpha \in E_6$.

Now, the structures of the groups $(E_6)^{\lambda\gamma}, (E_6)^\gamma, (E_6)^\sigma$ and $(E_6)^\lambda$ are as follows.

Theorem 3.3.1. [EI] *The group $(E_6)^{\lambda\gamma}$ is isomorphic to the group $Sp(4)/\mathbf{Z}_2$: $(E_6)^{\lambda\gamma} \cong (E_6)^{\tau\gamma} \cong Sp(4)/\mathbf{Z}_2, \mathbf{Z}_2 = \{E, -E\}$.*

Proof. We define a mapping $\varphi_{\text{EI}} : Sp(4) \rightarrow (E_6)^{\tau\gamma}$ by

$$\varphi_{\text{EI}}(P)X = g^{-1}(P(gX)P^*), \quad X \in \mathfrak{J}^C,$$

where $g : \mathfrak{J}^C \rightarrow \mathfrak{J}(4, \mathbf{H})_0^C$ is the C -linear isomorphism. This mapping induces the required isomorphism (see [10, Section 3.12]). \square

Remark. From $\lambda(\alpha) = \tau\alpha\tau$ and $\tau\gamma = \gamma\tau$, we see that $\lambda(\gamma\alpha\gamma) = \tau(\gamma\alpha\gamma)\tau$, $\alpha \in E_6$.

Theorem 3.3.2. [EII] *The group $(E_6)^\gamma$ is isomorphic to the group $(Sp(1) \times SU(6))/\mathbf{Z}_2$: $(E_6)^\gamma \cong (Sp(1) \times SU(6))/\mathbf{Z}_2$, $\mathbf{Z}_2 = \{(1, E), (-1, -E)\}$, where $SU(6) = \{U \in M(6, \mathbb{C}) \mid (\tau^t U)U = 1, \det U = 1\}$.*

Proof. We define a mapping $\varphi_{E_2} : Sp(1) \times SU(6) \rightarrow (E_6)^\gamma$ by

$$\varphi_{E_2}(p, U)(M + \mathbf{a}) = k_J^{-1}(U(k_J M)^t U) + pak^{-1}(\tau^t U), M + \mathbf{a} \in \mathfrak{J}(3, \mathbf{H})^C \oplus (\mathbf{H}^3)^C = \mathfrak{J}^C,$$

where both of $k_J : \mathfrak{J}(3, \mathbf{H})^C \rightarrow \mathfrak{S}(6, \mathbb{C})$ and $k : M(3, \mathbf{H})^C \rightarrow M(6, \mathbb{C})$ are the \mathbb{C} -linear isomorphisms. This mapping induces the required isomorphism (see [10, Section 3.11]). \square

Theorem 3.3.3. [EIII] *The group $(E_6)^\sigma$ is isomorphic to the group $(U(1) \times Spin(10))/\mathbf{Z}_4$: $(E_6)^\sigma \cong (U(1) \times Spin(10))/\mathbf{Z}_4$, $\mathbf{Z}_4 = \{(1, \phi(1)), (-1, \phi(-1)), (i, \phi(-i)), (-i, \phi(i))\}$.*

Proof. We define a mapping $\varphi_{E_3} : U(1) \times Spin(10) \rightarrow (E_6)^\sigma$ by

$$\varphi_{E_3}(\theta, \delta) = \phi_1(\theta)\delta,$$

where $\phi_1(\theta) : \mathfrak{J}^C \rightarrow \mathfrak{J}^C$ is the \mathbb{C} -linear mapping. This mapping induces the required isomorphism (see [10, Section 3.10]). \square

Theorem 3.3.4. [EIV] *The group $(E_6)^\lambda$ is isomorphic to the group $F_4 : (E_6)^\lambda = (E_6)^\tau \cong F_4$.*

Proof. From the explanation at the beginning of this section, we have $(E_6)^\lambda \cong (E_6)^\tau$, so by proving $(E_6)^\tau \cong F_4$ we have the required isomorphism (see [10, Section 3.7]). \square

3.4. Types EV, EVI and EVII. Let \mathfrak{P}^C be the Freudenthal \mathbb{C} -vector space. We define \mathbb{C} -linear transformations λ, γ, σ and ι of \mathfrak{P}^C by

$$\begin{aligned} \gamma(X, Y, \xi, \eta) &= (\gamma X, \gamma Y, \xi, \eta), \\ \sigma(X, Y, \xi, \eta) &= (\sigma X, \sigma Y, \xi, \eta), \\ \lambda(X, Y, \xi, \eta) &= (Y, -X, \eta, -\xi), \\ \iota(X, Y, \xi, \eta) &= (-iX, iY, -i\xi, i\eta), (X, Y, \xi, \eta) \in \mathfrak{P}^C, \end{aligned}$$

where $i \in \mathbb{C}$ and γ, σ of the right hand side are the same ones as $\gamma \in G_2 \subset F_4 \subset E_6, \sigma \in F_4 \subset E_6$. Then we have that $\gamma, \sigma, \lambda, \iota \in E_7$ and $\gamma^2 = \sigma^2 = 1, \lambda^2 = \iota^2 = -1$. Hence, as in E_6 , the group E_7 has involutive inner automorphisms $\tilde{\gamma}, \tilde{\sigma}$ induced by γ, σ : $\tilde{\gamma}(\alpha) = \gamma\alpha\gamma, \tilde{\sigma}(\alpha) = \sigma\alpha\sigma, \alpha \in E_7$. Moreover, since $-1 \in z(E_7)$ (the center of E_7), λ, ι induce involutive inner automorphisms $\tilde{\lambda}, \tilde{\iota}$ of E_7 : $\tilde{\lambda}(\alpha) = \lambda\alpha\lambda^{-1}, \tilde{\iota}(\alpha) = \iota\alpha\iota^{-1}, \alpha \in E_7$.

Now, the structures of the groups $(E_7)^{\lambda\gamma}, (E_7)^\gamma$ and $(E_7)^\iota$ are as follows.

Theorem 3.4.1. [EV] *The group $(E_7)^{\lambda\gamma}$ is isomorphic to the group $SU(8)/\mathbf{Z}_2 : (E_7)^{\lambda\gamma} \cong (E_7)^{\tau\gamma} \cong SU(8)/\mathbf{Z}_2, \mathbf{Z}_2 = \{E, -E\}$.*

Proof. We define a mapping $\varphi_{E_5} : SU(8) \rightarrow (E_7)^{\tau\gamma}$ by

$$\varphi_{E_5}(A)P = \chi^{-1}(A(\chi P)^t A), P \in \mathfrak{P}^C,$$

where $\chi : \mathfrak{P}^C \rightarrow \mathfrak{S}(8, \mathbb{C})^C$ is the \mathbb{C} -linear isomorphism. This mapping induces the required isomorphism (see [10, Section 4.12]). \square

Remark. Since $\lambda\gamma$ is conjugate to $\tau\gamma$ in E_6 , it is also in E_7 .

Theorem 3.4.2. [EVI] *The group $(E_7)^\gamma$ is isomorphic to the group $(SU(2) \times Spin(12))/\mathbf{Z}_2$: $(E_7)^\gamma \cong (E_7)^{-\sigma} = (E_7)^\sigma \cong (SU(2) \times Spin(12))/\mathbf{Z}_2, \mathbf{Z}_2 = \{(E, 1), (-E, -\sigma)\}$.*

Proof. We define a mapping $\varphi_{E_6} : SU(2) \times Spin(12) \rightarrow (E_7)^\sigma$ by

$$\varphi_{E_6}(A, \beta) = \phi_2(A)\beta,$$

where $\phi_2(A) : \mathbb{P}^C \rightarrow \mathbb{P}^C$ is the C -linear mapping. This mapping induces the required isomorphism (see [10, Section 4.11]). \square

Remark. As for the fact that γ is conjugate to $-\sigma$ in E_7 , see [12, Proposition 4.3.5 (3)].

Theorem 3.4.3. [EVII] *The group $(E_7)^t$ is isomorphic to the group $(U(1) \times E_6)/\mathbf{Z}_3$: $(E_7)^t \cong (U(1) \times E_6)/\mathbf{Z}_3$, $\mathbf{Z}_3 = \{(1, 1), (\omega, \phi(\omega^2)), (\omega^2, \phi(\omega))\}$, where $\omega \in C$, $\omega^3 = 1$, $\omega \neq 1$.*

Proof. We define a mapping $\varphi_{E_7} : U(1) \times E_6 \rightarrow (E_7)^t$ by

$$\varphi_{E_7}(\theta, \beta) = \phi(\theta)\beta,$$

where $\phi(\theta) : \mathbb{P}^C \rightarrow \mathbb{P}^C$ is the C -linear mapping. This mapping induces the required isomorphism (see [10, Section 4.10]). \square

3.5. Types EVIII and EIX. Let \mathfrak{e}_8^C be 248 dimensional C -vector space. We define C -linear transformations λ_ω, γ and ν of \mathfrak{e}_8^C by

$$\lambda_\omega(\Phi, P, Q, r, s, t) = (\lambda\Phi\lambda^{-1}, \lambda Q, -\lambda P, -r, -t, -s),$$

$$\gamma(\Phi, P, Q, r, s, t) = (\gamma\Phi\gamma, \gamma P, \gamma Q, r, s, t),$$

$$\nu(\Phi, P, Q, r, s, t) = (\Phi, -P, -Q, r, s, t), (\Phi, P, Q, r, s, t) \in \mathfrak{e}_8^C,$$

where λ, γ of right hand side are same ones $\lambda \in E_7, \gamma \in G_2 \subset F_4 \subset E_6 \subset E_7$. Then we have that $\lambda_\omega, \gamma, \nu \in E_8$ and $\lambda_\omega^2 = \gamma^2 = \nu^2 = 1$. Hence $\lambda_\omega, \gamma, \nu$ induce involutive inner automorphisms $\tilde{\lambda}_\omega, \tilde{\gamma}, \tilde{\nu}$ of E_8 : $\tilde{\lambda}_\omega(\alpha) = (\lambda_\omega)\alpha(\lambda_\omega), \tilde{\gamma}(\alpha) = \gamma\alpha\gamma, \tilde{\nu}(\alpha) = \nu\alpha\nu, \alpha \in E_8$. (Remark. λ_ω is nothing but $\lambda\lambda'$ defined in [10, Section 5.5].)

Now, the structures of the groups $(E_8)^{\lambda_\omega\gamma}$ and $(E_8)^\nu$ are as follows.

Theorem 3.5.1. [EVIII] *The group $(E_8)^{\lambda_\omega\gamma}$ is isomorphic to the group $Ss(16)$: $(E_8)^{\lambda_\omega\gamma} \cong Ss(16)$.*

Proof. Since the homomorphism between $(E_8)^{\lambda_\omega\gamma}$ and $Ss(16)$ is not found in E_8 defined in Section 2.5 until now, we omit this proof (see [10, Section 5.8]). \square

Theorem 3.5.2. [EIX] *The group $(E_8)^\nu$ is isomorphic to the group $(SU(2) \times E_7)/\mathbf{Z}_2$: $(E_8)^\nu \cong (SU(2) \times E_7)/\mathbf{Z}_2$, $\mathbf{Z}_2 = \{(E, 1), (-E, -1)\}$.*

Proof. We define a mapping $\varphi_{E_9} : SU(2) \times E_7 \rightarrow (E_7)^\nu$ by

$$\varphi_{E_9}(A, \beta) = \phi_3(A)\beta,$$

where $\phi_3(A) : \mathfrak{e}_8^C \rightarrow \mathfrak{e}_8^C$ is the C -linear transformation. This mapping induces the required isomorphism (see [10, Section 5.7]. ϕ_3 is nothing but φ_3 defined in [10, Theorem 5.7.4]). \square

4. GLOBALLY EXCEPTIONAL $\mathbb{Z}_2 \times \mathbb{Z}_2$ - SYMMETRIC SPACES

In this chapter, for $G = G_2, F_4, E_6, E_7$ or E_8 , we determine the type $(G/G^\sigma, G/G^\tau, G/G^{\sigma\tau})$ of globally exceptional $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space and the structure of group $G^\sigma \cap G^\tau$ by giving a pair of involutive inner automorphisms $\tilde{\sigma}$ and $\tilde{\tau}$ of G . Most of fundamental K -linear transformations and involutive automorphisms used later are defined in previous chapter, where $K = \mathbf{R}, C$, and others are defined each times.

Even if some proofs of this chapter are similar to ones of the preceding articles [5],[6],[7],[8],[10],[11],[12] and [13], we rewrite in detail again as much as possible. As mentioned in Tables 1,2, we also omit a sing \sim for the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

- $[G_2]$ We study one type in here.

4.1. Type G-G-G. In this section, we give a pair of involutive inner automorphisms $\tilde{\gamma}$ and $\tilde{\gamma}_H$, where $\tilde{\gamma}_H$ is induced by an \mathbf{R} -linear transformation γ_H defined below.

We define \mathbf{R} -linear transformations γ_H, γ_C of \mathbf{H} by

$$\begin{aligned}\gamma_H(a + be_2) &= a - be_2, \\ \gamma_C(a + be_2) &= \bar{a} + \bar{b}e_2, \quad a + be_2 \in \mathbf{C} \oplus \mathbf{C}e_2 = \mathbf{H}.\end{aligned}$$

Then γ_H, γ_C are naturally extended to \mathbf{R} -linear transformations γ_H, γ_C of \mathfrak{C} as follows:

$$\begin{aligned}\gamma_H(x + ye_4) &= \gamma_H x + (\gamma_H y)e_4, \\ \gamma_C(x + ye_4) &= \gamma_C x + (\gamma_C y)e_4, \quad x + ye_4 \in \mathbf{H} \oplus \mathbf{H}e_4 = \mathfrak{C}.\end{aligned}$$

Needless to say, we have that $\gamma_H, \gamma_C \in G_2$, $\gamma_H^2 = \gamma_C^2 = 1$, $\gamma\gamma_H = \gamma_H\gamma$, $\gamma\gamma_C = \gamma_C\gamma$. Hence γ_H, γ_C induce involutive inner automorphisms $\tilde{\gamma}_H, \tilde{\gamma}_C$ of G_2 : $\tilde{\gamma}_H(\alpha) = \gamma_H\alpha\gamma_H$, $\tilde{\gamma}_C(\alpha) = \gamma_C\alpha\gamma_C$, $\alpha \in G_2$.

Lemma 4.1.1. *In G_2 , we have the following facts.*

- (1) γ is conjugate to both of γ_H and $\gamma\gamma_H$: $\gamma \sim \gamma_H, \gamma \sim \gamma\gamma_H$.
- (2) γ is conjugate to both of γ_C and $\gamma\gamma_C$: $\gamma \sim \gamma_C, \gamma \sim \gamma\gamma_C$.

Proof. (1) We define \mathbf{R} -linear isomorphisms $\delta_1, \delta_2 : \mathfrak{C} \rightarrow \mathfrak{C}$ by

$$\begin{aligned}\delta_1 : 1 &\mapsto 1, e_1 \mapsto e_1, e_2 \mapsto e_4, e_3 \mapsto e_5, e_4 \mapsto e_2, e_5 \mapsto e_3, e_6 \mapsto -e_6, e_7 \mapsto -e_7, \\ \delta_2 : 1 &\mapsto 1, e_1 \mapsto e_1, e_2 \mapsto -e_6, e_3 \mapsto -e_7, e_4 \mapsto -e_4, e_5 \mapsto -e_5, e_6 \mapsto -e_2, e_7 \mapsto -e_3,\end{aligned}$$

where 1 and $e_k, k = 1, 2, \dots, 7$ are the basis of \mathfrak{C} , respectively. Then we see $\delta_1, \delta_2 \in G_2$, $\delta_1^2 = \delta_2^2 = 1$. Hence, by straightforward computation, we have $\delta_1\gamma = \gamma_H\delta_1, \delta_2\gamma = (\gamma\gamma_H)\delta_2$, that is, $\gamma \sim \gamma_H, \gamma \sim \gamma\gamma_H$ in G_2 .

(2) We define \mathbf{R} -linear transformations $\delta_3, \delta_4 : \mathfrak{C} \rightarrow \mathfrak{C}$ by

$$\begin{aligned}\delta_3 : 1 &\mapsto 1, e_1 \mapsto e_4, e_2 \mapsto e_2, e_3 \mapsto e_6, e_4 \mapsto e_1, e_5 \mapsto -e_5, e_6 \mapsto e_3, e_7 \mapsto -e_7, \\ \delta_4 : 1 &\mapsto 1, e_1 \mapsto e_5, e_2 \mapsto e_2, e_3 \mapsto -e_7, e_4 \mapsto -e_4, e_5 \mapsto e_1, e_6 \mapsto -e_6, e_7 \mapsto -e_3,\end{aligned}$$

respectively. Then as in (1) above, we have that $\delta_3, \delta_4 \in G_2$, $\delta_3^2 = \delta_4^2 = 1$, $\delta_3\gamma = \gamma_C\delta_3, \delta_4\gamma = (\gamma\gamma_C)\delta_4$, that is, $\gamma \sim \gamma_C, \gamma \sim \gamma\gamma_C$ in E_6 . \square

We have the following proposition which is the direct result of Lemma 4.1.1.

Proposition 4.1.2. *The group $(G_2)^\gamma$ is isomorphic to both of the groups $(G_2)^{\gamma_H}$ and $(G_2)^{\gamma\gamma_H}$: $(G_2)^\gamma \cong (G_2)^{\gamma_H} \cong (G_2)^{\gamma\gamma_H}$.*

From the result of type G in Table 2 and Proposition 4.1.2, we have the following theorem.

Theorem 4.1.3. *For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \gamma\} \times \{1, \gamma_H\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(G_2/(G_2)^\gamma, G_2/(G_2)^{\gamma_H}, G_2/(G_2)^{\gamma\gamma_H}) = (G_2/(G_2)^\gamma, G_2/(G_2)^\gamma, G_2/(G_2)^\gamma)$, that is, (G, G, G) , abbreviated as G.*

Here, we prove lemma needed and make some preparations for theorem below.

Lemma 4.1.4. *The mapping $\varphi_G : Sp(1) \times Sp(1) \rightarrow (G_2)^\gamma$ of Theorem 3.1.1 satisfies the equalities :*

- (1) $\gamma_H = \varphi_G(e_1, e_1), \gamma_C = \varphi_G(e_2, e_2)$.
- (2) $\gamma_H\varphi_G(p, q)\gamma_H = \varphi_G(\gamma_H p, \gamma_H q)$.

Proof. The proof of (1) is omitted (see [11, Lemma 1.3.3] in detail). The equality of (2) is the direct result of (1). \square

Consider a group $\mathcal{Z}_2 = \{1, \gamma_c\}$. Then the group $\mathcal{Z}_2 = \{1, \gamma_c\}$ acts on the group $U(1) \times U(1)$ by

$$\gamma_c(a, b) = (\gamma_c a, \gamma_c b)$$

and let $(U(1) \times U(1)) \rtimes \mathcal{Z}_2$ be the semi-direct product of $U(1) \times U(1)$ and \mathcal{Z}_2 with this action.

Now, we determine the structure of the group $(G_2)^\gamma \cap (G_2)^{\gamma_H}$.

Theorem 4.1.5. *We have that $(G_2)^\gamma \cap (G_2)^{\gamma_H} \cong (U(1) \times U(1))/\mathbb{Z}_2 \rtimes \mathcal{Z}_2$, $\mathbb{Z}_2 = \{(1, 1), (-1, -1)\}$, $\mathcal{Z}_2 = \{1, \gamma_c\}$.*

Proof. We define a mapping $\varphi_{415} : (U(1) \times U(1)) \rtimes \{1, \gamma_c\} \rightarrow (G_2)^\gamma \cap (G_2)^{\gamma_H}$ by

$$\begin{aligned} \varphi_{415}(a, b, 1) &= \varphi_G(a, b), \\ \varphi_{415}(a, b, \gamma_c) &= \varphi_G(a, b) \gamma_c, \end{aligned}$$

where φ_G is defined in Theorem 3.1.1. From $\gamma\gamma_c = \gamma_c\gamma$, $\gamma\gamma_H = \gamma_H\gamma$ and Lemma 4.1.4 (1), we have $\varphi_{415}(a, b, 1), \varphi_{415}(a, b, \gamma_c) \in (G_2)^\gamma \cap (G_2)^{\gamma_H}$. Hence φ_{415} is well-defined. Using $(ae_2)c = (a\bar{c})e_2$, $a, c \in U(1)$, we can confirm that φ_{415} is a homomorphism. Indeed, we show that the case of $\varphi_{415}(a, b, \gamma_c) \varphi_{415}(c, d, 1) = \varphi_{415}((a, b, \gamma_c)(c, d, 1))$ as example. For the left hand side of this equality, we have that

$$\begin{aligned} \varphi_{415}(a, b, \gamma_c) \varphi_{415}(c, d, 1) &= (\varphi_G(a, b) \gamma_c) \varphi_G(c, d) \\ &= (\varphi_G(a, b) \varphi_G(e_2, e_2)) \varphi_G(c, d) \\ &= \varphi_G((ae_2)c, (be_2)d) \\ &= \varphi_G((a\bar{c})e_2, (b\bar{d})e_2). \end{aligned}$$

On the other hand, for the right hand side of same one, we have that

$$\begin{aligned} \varphi_{415}((a, b, \gamma_c)(c, d, 1)) &= \varphi_{415}((a, b) \gamma_c(c, d), \gamma_c) \\ &= \varphi_G(a(\gamma_c c), b(\gamma_c d)) \gamma_c \\ &= \varphi_G(a\bar{c}, b\bar{d}) \varphi_G(e_2, e_2) \\ &= \varphi_G((a\bar{c})e_2, (b\bar{d})e_2) \end{aligned}$$

that is, $\varphi_{415}(a, b, \gamma_c) \varphi_{415}(c, d, 1) = \varphi_{415}((a, b, \gamma_c)(c, d, 1))$. Similarly, the other cases are shown.

We shall show that φ_{415} is surjection. Let $\alpha \in (G_2)^\gamma \cap (G_2)^{\gamma_H}$. Since $(G_2)^\gamma \cap (G_2)^{\gamma_H} \subset (G_2)^\gamma$, there exist $p, q \in Sp(1)$ such that $\alpha = \varphi_G(p, q)$ (Theorem 3.1.1). Moreover, from $\alpha = \varphi_G(p, q) \in (G_2)^{\gamma_H}$, that is, $\gamma_H \varphi_G(p, q) \gamma_H = \varphi_G(p, q)$, we have $\varphi_G(\gamma_H p, \gamma_H q) = \varphi_G(p, q)$ (Lemma 4.1.4 (2)). Hence it follows that

$$\begin{cases} \gamma_H p = p \\ \gamma_H q = q \end{cases} \quad \text{or} \quad \begin{cases} \gamma_H p = -p \\ \gamma_H q = -q. \end{cases}$$

In the former case, we see that $p, q \in U(1)$, then set $p = a, q = b, a, b \in U(1)$. Hence we have that $\alpha = \varphi_G(a, b) = \varphi_{415}(a, b, 1)$. In the latter case, we can find the explicit form of p, q as follow: $p = p_2 e_2 + p_3 e_3 = (p_2 + p_3 e_1) e_2, q = q_2 e_2 + q_3 e_3 = (q_2 + q_3 e_1) e_2, p_k, q_k \in \mathbf{R}, k = 2, 3$, that is, $p, q \in U(1) e_2 = \{u e_2 \mid u \in U(1)\}$. Then, set $p = a e_2, q = b e_2, a, b \in U(1)$, by using $\gamma_c = \varphi_G(e_2, e_2)$ (Lemma 4.1.4 (1)), we have that

$$\alpha = \varphi_G(a e_2, b e_2) = \varphi_G(a, b) \varphi_G(e_2, e_2) = \varphi_G(a, b) \gamma_c = \varphi_{415}(a, b, \gamma_c).$$

Thus φ_{415} is surjection.

From $\text{Ker } \varphi_G = \{(1, 1), (-1, -1)\}$, we can easily obtain that $\text{Ker } \varphi_{415} = \{(1, 1, 1), (-1, -1, 1)\} \cong (\mathbb{Z}_2, 1)$.

Therefore we have the required isomorphism

$$(G_2)^\gamma \cap (G_2)^{\gamma_H} \cong (U(1) \times U(1))/\mathbb{Z}_2 \rtimes \mathbb{Z}_2.$$

□

• **[F₄]** We study three types in here.

4.2. Type FI-I-I. In this section, we give a pair of involutive inner automorphisms $\tilde{\gamma}$ and $\tilde{\gamma}_H$.

We define \mathbf{R} -linear transformations γ_H, γ_c of \mathfrak{J} by

$$\gamma_H X = \begin{pmatrix} \xi_1 & \gamma_H x_3 & \overline{\gamma_H x_2} \\ \overline{\gamma_H x_3} & \xi_2 & \gamma_H x_1 \\ \gamma_H x_2 & \overline{\gamma_H x_1} & \xi_3 \end{pmatrix}, \quad \gamma_c X = \begin{pmatrix} \xi_1 & \gamma_c x_3 & \overline{\gamma_c x_2} \\ \overline{\gamma_c x_3} & \xi_2 & \gamma_c x_1 \\ \gamma_c x_2 & \overline{\gamma_c x_1} & \xi_3 \end{pmatrix}, \quad X \in \mathfrak{J},$$

where γ_H, γ_c of right hand side are the same ones as $\gamma_H, \gamma_c \in G_2$. Then we have that $\gamma_H, \gamma_c \in F_4, \gamma_H^2 = \gamma_c^2 = 1$. Hence γ_H, γ_c induce involutive inner automorphisms $\tilde{\gamma}_H, \tilde{\gamma}_c$ of F_4 : $\tilde{\gamma}_H(\alpha) = \gamma_H \alpha \gamma_H, \tilde{\gamma}_c(\alpha) = \gamma_c \alpha \gamma_c, \alpha \in F_4$. (Remark. In F_4 , we use γ_c , however we do not use $\tilde{\gamma}_c$.)

Moreover, using the inclusion $G_2 \subset F_4$, the \mathbf{R} -linear transformations δ_1, δ_2 defined in Lemma 4.1.1 are naturally extended to \mathbf{R} -linear transformations of \mathfrak{J} as follows:

$$\delta_k X = \begin{pmatrix} \xi_1 & \delta_k x_3 & \overline{\delta_k x_2} \\ \overline{\delta_k x_3} & \xi_2 & \delta_k x_1 \\ \delta_k x_2 & \overline{\delta_k x_1} & \xi_3 \end{pmatrix}, \quad X \in \mathfrak{J}, \quad k = 1, 2.$$

Then we see $\delta_1, \delta_2 \in F_4, \delta_1^2 = \delta_2^2 = 1$. As in G_2 , since we easily see that $\gamma \sim \gamma_H, \gamma \sim \gamma \gamma_H$ in F_4 , we have the following proposition.

Proposition 4.2.1. *The group $(F_4)^\gamma$ is isomorphic to both of the groups $(F_4)^{\gamma_H}$ and $(F_4)^{\gamma \gamma_H}$: $(F_4)^\gamma \cong (F_4)^{\gamma_H} \cong (F_4)^{\gamma \gamma_H}$.*

From the result of type FI in Table 2 and Proposition 4.2.1, we have the following theorem.

Theorem 4.2.2. *For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \gamma\} \times \{1, \gamma_H\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(F_4/(F_4)^\gamma, F_4/(F_4)^{\gamma_H}, F_4/(F_4)^{\gamma \gamma_H}) = (F_4/(F_4)^\gamma, F_4/(F_4)^\gamma, F_4/(F_4)^\gamma)$, that is, type (FI, FI, FI), abbreviated as FI-I-I.*

Here, we prove lemma needed and make some preparations for the theorem below.

Lemma 4.2.3. *The mapping $\varphi_{F1} : Sp(1) \times Sp(3) \rightarrow (F_4)^\gamma$ of Theorem 3.2.1 satisfies the equalities:*

$$(1) \quad \gamma_H = \varphi_{F1}(e_1, e_1 E), \quad \gamma_c = \varphi_{F1}(e_2, e_2 E). \\ (2) \quad \gamma_H \varphi_{F1}(p, A) \gamma_H = \varphi_{F1}(\gamma_H p, \gamma_H A).$$

Proof. The proof of (1) is omitted (see [11, Lemma 2.3.4] in detail). The equality of (2) is the direct result of (1). □

Consider a group $\mathbb{Z}_2 = \{1, \gamma_c\}$. Then the group $\mathbb{Z}_2 = \{1, \gamma_c\}$ acts on the group $U(1) \times U(3)$ by

$$\gamma_c(a, B) = (\gamma_c a, \gamma_c B)$$

and let $(U(1) \times U(3)) \rtimes \mathbb{Z}_2$ be the semi-direct product of $U(1) \times U(3)$ and \mathbb{Z}_2 with this action.

Now, we determine the structure of the group $(F_4)^\gamma \cap (F_4)^{\gamma_H}$.

Theorem 4.2.4. *We have that $(F_4)^\gamma \cap (F_4)^{\gamma_H} \cong (U(1) \times U(3))/\mathbb{Z}_2 \rtimes \mathbb{Z}_2, \mathbb{Z}_2 = \{(1, E), (-1, -E)\}, \mathbb{Z}_2 = \{1, \gamma_c\}$.*

Proof. We define a mapping $\varphi_{424} : (U(1) \times U(3)) \rtimes \{1, \gamma_c\} \rightarrow (F_4)^\gamma \cap (F_4)^{\gamma_H}$ by

$$\begin{aligned}\varphi_{424}(a, B, 1) &= \varphi_{F_1}(a, B), \\ \varphi_{424}(a, B, \gamma_c) &= \varphi_{F_1}(a, B)\gamma_c,\end{aligned}$$

where φ_{F_1} is defined in Theorem 3.2.1. As the proof of Theorem 4.1.5, it is easily to verify that φ_{424} is well-defined and a homomorphism.

We shall show that φ_{424} is surjection. Let $\alpha \in (F_4)^\gamma \cap (F_4)^{\gamma_H}$. Since $(F_4)^\gamma \cap (F_4)^{\gamma_H} \subset (F_4)^\gamma$, there exist $p \in Sp(1)$ and $A \in Sp(3)$ such that $\alpha = \varphi_{F_1}(p, A)$ (Theorem 3.2.1). Moreover, from $\alpha = \varphi_{F_1}(p, A) \in (F_4)^{\gamma_H}$, that is, $\gamma_H \varphi_{F_1}(p, A) \gamma_H = \varphi_{F_1}(p, A)$, we have $\varphi_{F_1}(\gamma_H p, \gamma_H A) = \varphi_{F_1}(p, A)$ (Lemma 4.2.3 (2)). Hence it follows that

$$\begin{cases} \gamma_H p = p \\ \gamma_H A = A \end{cases} \quad \text{or} \quad \begin{cases} \gamma_H p = -p \\ \gamma_H A = -A. \end{cases}$$

In the former case, we easily see that $p \in U(1)$, $A \in U(3)$, then set $p = a \in U(1)$ and $A = B \in U(3)$. Hence we have that $\alpha = \varphi_{F_1}(a, B) = \varphi_{424}(a, B, 1)$. In the latter case, in the way similar to the former case of Theorem 4.1.5 we find that $p \in U(1)e_2 = \{ue_2 \mid u \in U(1)\}$, $A \in U(3)(e_2E) = \{B(e_2E) \mid B \in U(3)\}$. Then, set $p = ae_2$, $A = B(e_2E)$, $a \in U(1)$, $B \in U(3)$, by using $\gamma_c = \varphi_{F_1}(e_2, e_2E)$ (Lemma 4.2.3 (1)), we have that

$$\alpha = \varphi_{F_1}(ae_2, B(e_2E)) = \varphi_{F_1}(a, B)\varphi_{F_1}(e_2, e_2E) = \varphi_{F_1}(a, B)\gamma_c = \varphi_{424}(a, B, \gamma_c).$$

Thus φ_{424} is surjection.

From $\text{Ker } \varphi_{F_1} = \{(1, E), (-1, -E)\}$, we can easily obtain that $\text{Ker } \varphi_{424} = \{(1, E, 1), (-1, -E, 1)\} \cong (\mathbb{Z}_2, 1)$.

Therefore we have the required isomorphism

$$(F_4)^\gamma \cap (F_4)^{\gamma_H} \cong (U(1) \times U(3))/\mathbb{Z}_2 \rtimes \mathbb{Z}_2.$$

□

4.3. Type FI-I-II. In this section, we give a pair of involutive inner automorphisms $\tilde{\gamma}$ and $\tilde{\gamma}\sigma$

Lemma 4.3.1. *In F_4 , γ is conjugate to $\gamma\sigma$: $\gamma \sim \gamma\sigma$.*

Proof. We define an \mathbf{R} -linear transformation δ_5 of \mathfrak{J} by

$$\delta_5 X = \begin{pmatrix} \xi_1 & x_3 e_4 & \bar{x}_2 e_4 \\ -e_4 \bar{x}_3 & \xi_2 & -e_4 x_1 e_4 \\ -e_4 x_2 & -e_4 \bar{x}_1 e_4 & \xi_3 \end{pmatrix}, \quad X \in \mathfrak{J}.$$

Then we have that $\delta_5 \in F_4$, $\delta_5^2 = 1$, $\delta_5 \gamma = (\gamma\sigma)\delta_5$, that is, $\gamma \sim \gamma\sigma$ in F_4 . □

We have the following proposition which is the direct result of Lemma 4.3.1.

Proposition 4.3.2. *The group $(F_4)^\gamma$ is isomorphic to the group $(F_4)^{\gamma\sigma}$: $(F_4)^\gamma \cong (F_4)^{\gamma\sigma}$.*

From the result of types FI, FII in Table 2 and Proposition 4.3.2, we have the following theorem.

Theorem 4.3.3. *For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \gamma\} \times \{1, \gamma\sigma\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(F_4/(F_4)^\gamma, F_4/(F_4)^{\gamma\sigma}, F_4/(F_4)^{\gamma(\gamma\sigma)}) = (F_4/(F_4)^\gamma, F_4/(F_4)^\gamma, F_4/(F_4)^\sigma)$, that is, type (FI, FI, FII), abbreviated as FI-I-II.*

Here, we prove lemma needed in theorem below.

Lemma 4.3.4. *The mapping $\varphi_{\text{FI}} : Sp(1) \times Sp(3) \rightarrow (F_4)^\gamma$ of Theorem 3.2.1 satisfies the equalities:*

- (1) $\gamma = \varphi_{\text{FI}}(-1, -E)$, $\sigma = \varphi_{\text{FI}}(-1, I_1)$.
- (2) $\gamma\varphi_{\text{FI}}(p, A)\gamma = \varphi_{\text{FI}}(p, A)$, $\sigma\varphi_{\text{FI}}(p, A)\sigma = \varphi_{\text{FI}}(p, I_1AI_1)$,

where $I_1 = \text{diag}(-1, 1, 1)$.

Proof. The proof of (1) is omitted (see [11, Lemma 2.3.4] in detail). The equalities of (2) are the direct result of (1). \square

Now, we determine the structure of the group $(F_4)^\gamma \cap (F_4)^{\gamma\sigma}$.

Theorem 4.3.5. *We have that $(F_4)^\gamma \cap (F_4)^{\gamma\sigma} \cong (Sp(1) \times Sp(1) \times Sp(2))/\mathbf{Z}_2$, $\mathbf{Z}_2 = \{(1, 1, E), (-1, -1, E)\}$.*

Proof. We define a mapping $\varphi_{435} : Sp(1) \times Sp(1) \times Sp(2) \rightarrow (F_4)^\gamma \cap (F_4)^{\gamma\sigma}$ by

$$\varphi_{435}(p, q, B) = \varphi_{\text{FI}}(p, h(q, B)),$$

where h is defined by $h : Sp(1) \times Sp(2) \rightarrow Sp(3)$, $h(q, B) = \begin{pmatrix} q & 0 \\ 0 & B \end{pmatrix}$. Since the mapping φ_{435} is the restriction of the mapping φ_{FI} , it is easily to verify that φ_{435} is well-defined and a homomorphism.

We shall show that φ_{435} is surjection. Let $\alpha \in (F_4)^\gamma \cap (F_4)^{\gamma\sigma}$. Since $(F_4)^\gamma \cap (F_4)^{\gamma\sigma} \subset (F_4)^\gamma$, there exist $p \in Sp(1)$ and $A \in Sp(3)$ such that $\alpha = \varphi_{\text{FI}}(p, A)$ (Theorem 3.2.1). Moreover, from $\alpha = \varphi_{\text{FI}}(p, A) \in (F_4)^{\gamma\sigma}$, that is, $(\gamma\sigma)\varphi_{\text{FI}}(p, A)(\sigma\gamma) = \varphi_{\text{FI}}(p, A)$, using $\gamma\varphi_{\text{FI}}(p, A)\gamma = \varphi_{\text{FI}}(p, A)$ and $\sigma\varphi_{\text{FI}}(p, A)\sigma = \varphi_{\text{FI}}(p, I_1AI_1)$ (Lemma 4.3.4 (2)), we have $\varphi_{\text{FI}}(p, I_1AI_1) = \varphi_{\text{FI}}(p, A)$. Hence it follows that

$$\begin{cases} p = p \\ I_1AI_1 = A \end{cases} \quad \text{or} \quad \begin{cases} p = -p \\ I_1AI_1 = -A. \end{cases}$$

In the former case, it is trivial that $p \in Sp(1)$, and we get the explicit form of $A \in Sp(3)$ as follows:

$$A = \begin{pmatrix} q & 0 \\ 0 & B \end{pmatrix}, q \in Sp(1), B \in Sp(2).$$

Hence we have $\alpha = \varphi_{\text{FI}}(p, h(q, B)) = \varphi_{435}(p, q, B)$. In the latter case, this case is impossible because of $p = 0$ for $p \in Sp(1)$. Thus φ_{435} is surjection.

From $\text{Ker } \varphi_{\text{FI}} = \{(1, E), (-1, -E)\}$, we can easily obtain that $\text{Ker } \varphi_{435} = \{(1, 1, E), (-1, -1, -E)\} \cong \mathbf{Z}_2$.

Therefore we have the required isomorphism

$$(F_4)^\gamma \cap (F_4)^{\gamma\sigma} \cong (Sp(1) \times Sp(1) \times Sp(2))/\mathbf{Z}_2.$$

\square

4.4. Type FII-II-II. In this section, we give a pair of involutive inner automorphisms $\tilde{\sigma}$ and $\tilde{\sigma}'$, where an \mathbf{R} -linear transformation σ' of \mathfrak{J} is defined below.

We define an \mathbf{R} -linear transformation σ' of \mathfrak{J} by

$$\sigma'X = \begin{pmatrix} \xi_1 & x_3 & -\bar{x}_2 \\ \bar{x}_3 & \xi_2 & -x_1 \\ -x_2 & -\bar{x}_1 & \xi_3 \end{pmatrix}, X \in \mathfrak{J}.$$

Then we have that $\sigma' \in F_4$, $\sigma'^2 = 1$, $\sigma\sigma' = \sigma'\sigma$. Hence σ' induces involutive inner automorphism $\tilde{\sigma}'$ of F_4 : $\tilde{\sigma}'(\alpha) = \sigma'\alpha\sigma'$, $\alpha \in F_4$.

Lemma 4.4.1. *In F_4 , σ is conjugate to both of σ' and $\sigma\sigma'$: $\sigma \sim \sigma'$, $\sigma \sim \sigma\sigma'$.*

Proof. We define \mathbf{R} -linear transformations δ_6, δ_7 of \mathfrak{J} by

$$\delta_6 X = \begin{pmatrix} \xi_3 & \bar{x}_1 & x_2 \\ x_1 & \xi_2 & \bar{x}_3 \\ \bar{x}_2 & x_3 & \xi_1 \end{pmatrix}, \quad \delta_7 X = \begin{pmatrix} \xi_2 & \bar{x}_3 & x_1 \\ x_3 & \xi_1 & \bar{x}_2 \\ \bar{x}_1 & x_2 & \xi_3 \end{pmatrix}, \quad X \in \mathfrak{J}.$$

Then we have that $\delta_6, \delta_7 \in F_4, \delta_6^2 = \delta_7^2 = 1$. Hence, by straightforward computation, we have that $\delta_6 \sigma = \sigma' \delta_6, \delta_7 \sigma = (\sigma \sigma') \delta_7$, that is, $\sigma \sim \sigma', \sigma \sim \sigma \sigma'$ in F_4 . \square

We have the following proposition which is the direct result of Lemma 4.4.1.

Proposition 4.4.2. *The group $(F_4)^\sigma$ is isomorphic to both of the groups $(F_4)^{\sigma'}$ and $(F_4)^{\sigma \sigma'}$: $(F_4)^\sigma \cong (F_4)^{\sigma'} \cong (F_4)^{\sigma \sigma'}$.*

From the result of type FII in Table 2 and Proposition 4.4.2, we have the following theorem.

Theorem 4.4.3. *For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \sigma\} \times \{1, \sigma'\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(F_4/(F_4)^\sigma, F_4/(F_4)^{\sigma'}, F_4/(F_4)^{\sigma \sigma'}) = (F_4/(F_4)^\sigma, F_4/(F_4)^\sigma, F_4/(F_4)^\sigma)$, that is, type (FII, FII, FII), abbreviated as FII-II-II.*

Here, we prove lemma needed in the theorem below.

Lemma 4.4.4. *The Lie algebra $(\mathfrak{f}_4)^\sigma \cap (\mathfrak{f}_4)^{\sigma'}$ of the group $(F_4)^\sigma \cap (F_4)^{\sigma'}$ is given by*

$$(\mathfrak{f}_4)^\sigma \cap (\mathfrak{f}_4)^{\sigma'} = \{D \in \mathfrak{so}(8)\} = \mathfrak{so}(8).$$

In particular, we have

$$\dim((\mathfrak{f}_4)^\sigma \cap (\mathfrak{f}_4)^{\sigma'}) = 28.$$

Proof. Since any element δ of the Lie algebra \mathfrak{f}_4 of the group F_4 is uniquely expressed as

$$\delta = D + \tilde{A}_1(a_1) + \tilde{A}_2(a_2) + \tilde{A}_3(a_3), \quad D \in \mathfrak{so}(8), a_i \in \mathbb{C}, k = 1, 2, 3,$$

using

$$\begin{aligned} \sigma \delta \sigma &= D + \tilde{A}_1(a_1) + \tilde{A}_2(-a_2) + \tilde{A}_3(-a_3), \\ \sigma' \delta \sigma' &= D + \tilde{A}_1(-a_1) + \tilde{A}_2(-a_2) + \tilde{A}_3(a_3), \end{aligned}$$

we can easily prove this lemma. \square

Now, we determine the structure of the group $(F_4)^\sigma \cap (F_4)^{\sigma'}$.

Theorem 4.4.5. *We have that $(F_4)^\sigma \cap (F_4)^{\sigma'} \cong Spin(8)$.*

Proof. Let $Spin(8) = \{(\alpha_1, \alpha_2, \alpha_3) \in SO(8) \times SO(8) \times SO(8) \mid (\alpha_1 x)(\alpha_2 y) = \overline{\alpha_3(\bar{x} \bar{y})}, x, y \in \mathbb{C}\}$.

We define a mapping $\varphi_{445} : Spin(8) \rightarrow (F_4)^\sigma \cap (F_4)^{\sigma'}$ by

$$\varphi_{445}(\alpha_1, \alpha_2, \alpha_3)X = \begin{pmatrix} \xi_1 & \alpha_3 x_3 & \overline{\alpha_2 x_2} \\ \overline{\alpha_3 x_3} & \xi_2 & \alpha_1 x_1 \\ \alpha_2 x_2 & \overline{\alpha_1 x_1} & \xi_3 \end{pmatrix}, \quad X \in \mathfrak{J}.$$

It is easily to verify that φ_{445} is well-defined, a homomorphism and injection.

We shall show that φ_{445} is surjection. From $(F_4)^\sigma \cong Spin(9)$ ([5, Proposition 1.4]), we have that $(F_4)^\sigma \cap (F_4)^{\sigma'} = ((F_4)^\sigma)^{\sigma'} \cong (Spin(9))^{\sigma'}$. Hence $(F_4)^\sigma \cap (F_4)^{\sigma'}$ is connected. Moreover, together with $\dim((\mathfrak{f}_4)^\sigma \cap (\mathfrak{f}_4)^{\sigma'}) = 28 = \dim(\mathfrak{so}(8))$ (Lemma 4.4.4), we have that φ_{445} is surjection.

Therefore we have the required isomorphism

$$(F_4)^\sigma \cap (F_4)^{\sigma'} \cong Spin(8).$$

\square

- [E₆] We study eight types in here.

4.5. Type EI-I-II. In this section, we give a pair of involutive automorphisms $\lambda\tilde{\gamma}$ and $\lambda\gamma\tilde{\gamma}_c$.

Let the C -linear transformations γ_H, γ_C of \mathfrak{J}^C be the complexification of $\gamma_H, \gamma_C \in G_2 \subset F_4$. Then we have that $\gamma_H, \gamma_C \in E_6, \gamma_H^2 = \gamma_C^2 = 1$, so γ_C involutive inner automorphism $\tilde{\gamma}_c$ of E_6 : $\tilde{\gamma}_c(\alpha) = \gamma_C \alpha \gamma_C, \alpha \in E_6$.

Using the inclusion $G_2 \subset F_4 \subset E_6$, the R -linear transformations δ_3, δ_4 defined in Lemma 4.1.1 are naturally extended to C -linear transformation of \mathfrak{J}^C . Hence, as in G_2 , since we easily see that $\delta_3\gamma = \gamma_C\delta_3, \delta_4\gamma = \gamma\gamma_C\delta_4$ as $\delta_3, \delta_4 \in E_6$, that is, $\gamma \sim \gamma_C, \gamma \sim \gamma\gamma_C$ in E_6 , we have the following proposition.

Proposition 4.5.1. (1) *The group $(E_6)^{\lambda\gamma}$ is isomorphic to the group $(E_6)^{\lambda\gamma\gamma_C}$: $(E_6)^{\lambda\gamma} \cong (E_6)^{\lambda\gamma\gamma_C}$.*

(2) *The group $(E_6)^\gamma$ is isomorphic to the group $(E_6)^{\gamma_C}$: $(E_6)^\gamma \cong (E_6)^{\gamma_C}$.*

Proof. (1) We define a mapping $f : (E_6)^{\lambda\gamma} \rightarrow (E_6)^{\lambda\gamma\gamma_C}$ by

$$f(\alpha) = \delta_4 \alpha \delta_4.$$

In order to prove this proposition, it is sufficient to show that the mapping f is well-defined. Indeed, it follows from $\gamma \sim \gamma\gamma_C$ that

$$\begin{aligned} \lambda(\gamma\gamma_C f(\alpha)\gamma_C\gamma) &= {}^t(\gamma\gamma_C(\delta_4\alpha\delta_4)\gamma_C\gamma)^{-1} = \gamma\gamma_C(\delta_4{}^t\alpha^{-1}\delta_4)\gamma_C\gamma \\ &= \delta_4(\gamma\lambda(\alpha)\gamma)\delta_4 = \delta_4\alpha\delta_4 = f(\alpha), \end{aligned}$$

that is, $f(\alpha) \in (E_6)^{\lambda\gamma\gamma_C}$.

(2) This isomorphism is the direct result of $\gamma \sim \gamma_C$. □

From the result of types EI, EII in Table 2 and Propositions 4.5.2, we have the following theorem.

Theorem 4.5.2. *For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \lambda\gamma\} \times \{1, \lambda\gamma\gamma_C\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(E_6/(E_6)^{\lambda\gamma}, E_6/(E_6)^{\lambda\gamma\gamma_C}, E_6/(E_6)^{(\lambda\gamma)(\lambda\gamma\gamma_C)}) = (E_6/(E_6)^{\lambda\gamma}, E_6/(E_6)^{\lambda\gamma\gamma_C}, E_6/(E_6)^{\gamma_C}) = (E_6/(E_6)^{\lambda\gamma}, E_6/(E_6)^{\lambda\gamma}, E_6/(E_6)^\gamma)$, that is, type (EI, EI, EII), abbreviated as EI-I-II.*

Here, we prove lemma and proposition needed and make some preparations for the theorem below.

Lemma 4.5.3. *The mapping $\varphi_{E_2} : Sp(1) \times SU(6) \rightarrow (E_6)^\gamma$ of Theorem 3.3.2 satisfies the following equalities:*

- (1) $\gamma = \varphi_{E_2}(-1, E), \gamma_H = \varphi_{E_2}(e_1, iI), \gamma_C = \varphi_{E_2}(e_2, J), \sigma = \varphi_{E_2}(-1, I_2).$
- (2) $\gamma\varphi_{E_2}(p, U)\gamma = \varphi_{E_2}(p, U), \gamma_H\varphi_{E_2}(p, U)\gamma_H = \varphi_{E_2}(\gamma_H p, IUI),$
 $\gamma_C\varphi_{E_2}(p, U)\gamma_C = \varphi_{E_2}(\gamma_C p, -JUI), \sigma\varphi_{E_2}(p, U)\sigma = \varphi_{E_2}(p, I_2UI_2).$
- (3) $\lambda(\varphi_{E_2}(p, U)) = \varphi_{E_2}(p, -J(\tau U)J),$

where $i \in C, I = \text{diag}(1, -1, 1, -1, 1, -1), J = \text{diag}(J_1, J_1, J_1), J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, I_2 = \text{diag}(-1, -1, 1, 1, 1, 1).$

Proof. The proof of (1) is omitted (see [11, Lemmas 3.5.7, 3.5.10] in detail). The equalities of (2) are the direct results of (1).

(3) Using the equality $\tau k(M) = -Jk(\tau M)J$, we have the required result. Indeed,

$$\begin{aligned}
 \lambda(\varphi_{\mathbb{E}_2}(p, U))(M + a) &= \tau(\varphi_{\mathbb{E}_2}(p, U))\tau(M + a) \text{ (see Section 3.3)} \\
 &= \tau(\varphi_{\mathbb{E}_2}(p, U))(\tau M + \tau a) \\
 &= \tau(k_J^{-1}(Uk_J(\tau M)^t U) + p(\tau a)k^{-1}(\tau^t U)) \\
 &= -\tau k^{-1}((Uk_J(\tau M)^t U)J) + p\tau k^{-1}(\tau^t U) \\
 &= k^{-1}(J(\tau U)J(kM)^t U)(-E)) + pak^{-1}(\tau^t(-J(\tau U)J)) \\
 &= k^{-1}((-J(\tau U)J)(kM)J(-J(\tau^t U)J)(-J)) + pak^{-1}(\tau^t(-J(\tau U)J)) \\
 &= k_J^{-1}((-J(\tau U)J)(k_J M)^t((-J(\tau U)J))) + pak^{-1}(\tau^t(-J(\tau U)J)) \\
 &= \varphi_{\mathbb{E}_2}(p, -J(\tau U)J)(M + a),
 \end{aligned}$$

that is, $\lambda(\varphi_{\mathbb{E}_2}(p, U)) = \varphi_{\mathbb{E}_2}(p, -J(\tau U)J)$. \square

Proposition 4.5.4. *The group $(E_6)^{\lambda\gamma} \cap (E_6)^{\lambda\gamma\gamma_c}$ is isomorphic to the group $(E_6)^{\lambda\gamma} \cap (E_6)^{\gamma_c}$: $(E_6)^{\lambda\gamma} \cap (E_6)^{\lambda\gamma\gamma_c} \cong (E_6)^{\lambda\gamma} \cap (E_6)^{\gamma_c}$.*

Proof. We define a mapping $g : (E_6)^{\lambda\gamma} \cap (E_6)^{\lambda\gamma\gamma_c} \rightarrow (E_6)^{\lambda\gamma} \cap (E_6)^{\gamma_c}$ by

$$g(\alpha) = \lambda(\alpha).$$

In order to prove this proposition, it is sufficient to show that the mapping g is well-defined. Indeed, it follows from $\lambda(\gamma) = \gamma$, $\lambda(\gamma_c) = \gamma_c$ that

$$\begin{aligned}
 \lambda(\gamma g(\alpha)\gamma) &= \lambda(\gamma\lambda(\alpha)\gamma) = \lambda(\alpha) = g(\alpha) \text{ and} \\
 \gamma_c g(\alpha)\gamma_c &= \gamma_c \lambda(\alpha)\gamma_c = \gamma(\gamma\gamma_c \lambda(\alpha)\gamma_c \gamma)\gamma = \gamma\alpha\gamma = \gamma(\gamma\lambda(\alpha)\gamma)\gamma = \lambda(\alpha) = g(\alpha),
 \end{aligned}$$

that is, $g(\alpha) \in (E_6)^{\lambda\gamma}$ and $g(\alpha) \in (E_6)^{\gamma_c}$. \square

Let $\{a = x + ye_2 \mid \bar{a}a = 1, x, y \in \mathbf{R}\} \subset Sp(1)$ be a group which is isomorphic to the ordinary unitary group $U(1)$, so this group is also denoted by $U(1)$. In this section, we use this as $U(1)$.

Consider a group $\mathcal{Z}_2 = \{1, \gamma_H\}$. Then the group $\mathcal{Z}_2 = \{1, \gamma_H\}$ acts on the group $U(1) \times SO(6)$ by

$$\gamma_H(a, A) = (\bar{a}, (iI)A(iI)^{-1}),$$

where $I = \text{diag}(1, -1, 1, -1, 1, -1)$, and let $(U(1) \times SO(6)) \rtimes \mathcal{Z}_2$ be the semi-direct product of $U(1) \times SO(6)$ and \mathcal{Z}_2 with this action.

Now, we determine the structure of the group $(E_6)^{\lambda\gamma} \cap (E_6)^{\lambda\gamma\gamma_c}$.

Theorem 4.5.5. *We have that $(E_6)^{\lambda\gamma} \cap (E_6)^{\lambda\gamma\gamma_c} \cong (U(1) \times SO(6))/\mathcal{Z}_2 \rtimes \mathcal{Z}_2$, $\mathcal{Z}_2 = \{(1, E), (-1, -E)\}$, $\mathcal{Z}_2 = \{1, \gamma_H\}$.*

Proof. We define a mapping $(U(1) \times SO(6)) \rtimes \{1, \gamma_H\} \rightarrow (E_6)^{\lambda\gamma} \cap (E_6)^{\gamma_c}$ by

$$\begin{aligned}
 \varphi_{456}((a, A), 1) &= \delta_7 \varphi_{\mathbb{E}_2}(a, A) \delta_7, \\
 \varphi_{456}((a, A), \gamma_H) &= \delta_7 (\varphi_{\mathbb{E}_2}(a, A) \gamma_H) \delta_7,
 \end{aligned}$$

where $\varphi_{\mathbb{E}_2}, \delta_7$ are defined in Theorem 3.3.2, Lemma 4.5.1, respectively. From Lemmas 4.5.1, 4.5.4, we have $\varphi_{456}((a, A), 1), \varphi_{456}((a, A), \gamma_H) \in (E_6)^{\lambda\gamma} \cap (E_6)^{\gamma_c}$. Hence φ_{456} is well-defined. Using $\gamma_H = \varphi_{\mathbb{E}_2}(e_1, iI)$ (Lemma 4.5.4 (1)), we can confirm that φ_{456} is a homomorphism. Indeed, we show that the case of $\varphi_{456}((a, A), \gamma_H) \varphi_{456}((b, B), 1) = \varphi_{456}(((a, A), \gamma_H)((b, B), 1))$ as example. For the left hand side of this equality, we have that

$$\begin{aligned}
 \varphi_{456}((a, A), \gamma_H) \varphi_{456}((b, B), 1) &= (\delta_7 (\varphi_{\mathbb{E}_2}(a, A) \gamma_H) \delta_7) (\delta_7 \varphi_{\mathbb{E}_2}(b, B) \delta_7) \\
 &= (\delta_7 (\varphi_{\mathbb{E}_2}(a, A) \varphi_{\mathbb{E}_2}(e_1, iI)) \delta_7) (\delta_7 \varphi_{\mathbb{E}_2}(b, B) \delta_7) \\
 &= \delta_7 (\varphi_{\mathbb{E}_2}(a \bar{b}, A(iI)B(iI)^{-1}) \gamma_H) \delta_7.
 \end{aligned}$$

On the other hand, for the right hand side of same one, we have that

$$\begin{aligned}\varphi_{456}(((a, A), \gamma_H)((b, B), 1)) &= \varphi_{456}((a, A)\gamma_H(b, B), \gamma_H) \\ &= \varphi_{456}((a\bar{b}, A(iI)B(iI)^{-1}), \gamma_H) \\ &= \delta_7(\varphi_{E_2}(a\bar{b}, A(iI)B(iI)^{-1})\gamma_H)\delta_7,\end{aligned}$$

that is, $\varphi_{456}((a, A), \gamma_H)\varphi_{456}((b, B), 1) = \varphi_{456}(((a, A), \gamma_H)((b, B), 1))$. Similarly, the other cases are shown.

We shall show that φ_{456} is surjection. Let $\alpha \in (E_6)^{\lambda\gamma} \cap (E_6)^{\gamma c}$. Hence, since $\alpha \in (E_6)^{\lambda\gamma} \cap (E_6)^{\gamma c} \subset (E_6)^{\gamma c} \cong (E_6)^{\gamma}$ (Proposition 4.5.2 (2)), there exist $p \in Sp(1)$ and $U \in SU(6)$ such that $\alpha = \delta_7\varphi_{E_2}(p, U)\delta_7$ (Theorem 3.3.2). Moreover, from $\alpha = \delta_7\varphi_{E_2}(p, U)\delta_7 \in (E_6)^{\lambda\gamma}$, that is, $\lambda(\gamma(\delta_7\varphi_{E_2}(p, U)\delta_7)\gamma) = \delta_7\varphi_{E_2}(p, U)\delta_7$, using $\gamma_c\varphi_{E_2}(p, U)\gamma_c = \varphi_{E_2}(\gamma_cp, -JUJ)$ and $\lambda(\varphi_{E_2}(p, U)) = \varphi_{E_2}(p, -J(\tau U)J)$ (Lemma 4.5.4 (2), (3)), we have that $\varphi_{E_2}(\gamma_cp, \tau A) = \varphi_{E_2}(p, A)$. Hence it follows that

$$\begin{cases} \gamma_cp = p \\ \tau U = U \end{cases} \quad \text{or} \quad \begin{cases} \gamma_cp = -p \\ \tau U = -U. \end{cases}$$

In the former case, we see that $p \in U(1) = \{a = p_0 + p_2e_2 \mid a\bar{a} = 1\}$ and $U \in SO(6)$. Hence we have that $\alpha = \delta_7\varphi_{E_2}(a, A)\delta_7 = \varphi_{456}((a, A), 1)$. In the latter case, we can find the explicit forms of $p \in Sp(1)$, $U \in SU(6)$ as follows:

$$p = p_1e_1 + p_2e_3 = ae_1 \ (a = p_1 - p_2e_2 \in U(1)), \quad U = A(iI), A \in SO(6).$$

Hence we have that

$$\begin{aligned}\alpha &= \delta_7(\varphi_{E_2}(ae_1, A(iI)))\delta_7 = \delta_7(\varphi_{E_2}(a, A)\varphi_{E_2}(e_1, iI))\delta_7 \\ &= \delta_7(\varphi_{E_2}(a, A)\gamma_H)\delta_7 = \varphi_{456}((a, A), \gamma_H).\end{aligned}$$

Thus φ_{456} is surjection.

From $\text{Ker } \varphi_{E_2} = \{(1, E), (-1, -E)\}$, we can easily obtain that $\text{Ker } \varphi_{456} = \{((1, E), 1), ((-1, -E), 1)\} \cong (\mathbf{Z}_2, 1)$. Therefore we have the following isomorphism $(E_6)^{\lambda\gamma} \cap (E_6)^{\gamma c} \cong (U(1) \times SO(6))/\mathbf{Z}_2 \rtimes \{1, \gamma_H\}$.

Namely, from Proposition 4.5.5, we have the required isomorphism

$$(E_6)^{\lambda\gamma} \cap (E_6)^{\lambda\gamma c} \cong (U(1) \times SO(6))/\mathbf{Z}_2 \rtimes \mathbf{Z}_2.$$

□

4.6. Type EI-I-III. In this section, we give a pair of involutive automorphisms $\lambda\tilde{\gamma}$ and $\lambda\gamma\tilde{\sigma}$.

Using the inclusion $F_4 \subset E_6$, the \mathbf{R} -linear transformation δ_5 defined in Lemma 4.3.1 is naturally extended to \mathbf{C} -linear transformation of \mathfrak{F}^C . Hence, as in F_4 , we easily see that $\delta_5\gamma = (\gamma\sigma)\delta_5$ as $\delta_5 \in F_4 \subset E_6$, that is, $\gamma \sim \gamma\sigma$ in E_6 .

Proposition 4.6.1. *The group $(E_6)^{\lambda\gamma}$ is isomorphic to the group $(E_6)^{\lambda\gamma\sigma} : (E_6)^{\lambda\gamma} \cong (E_6)^{\lambda\gamma\sigma}$.*

Proof. We define a mapping $f : (E_6)^{\lambda\gamma} \rightarrow (E_6)^{\lambda\gamma\sigma}$ by

$$f(\alpha) = \delta_5\alpha\delta_5^{-1},$$

where δ_5 is same one above. (Remark. since $\delta_5 \in F_4$, it follows that $\lambda(\delta_5) = \delta_5$.) In order to prove this proposition, it is sufficient to show that the mapping f is well-defined. However, it is almost evident from $\lambda(\delta_5) = \delta_5$ and $\delta_5\gamma = (\gamma\sigma)\delta_5$. □

From the result of types EI, EIII in Table 2 and Proposition 4.6.1, we have the following theorem.

Theorem 4.6.2. For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \lambda\gamma\} \times \{1, \lambda\gamma\sigma\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(E_6/(E_6)^{\lambda\gamma}, E_6/(E_6)^{\lambda\gamma\sigma}, E_6/(E_6)^{(\lambda\gamma)(\lambda\gamma\sigma)}) = (E_6/(E_6)^{\lambda\gamma}, E_6/(E_6)^{\lambda\gamma}, E_6/(E_6)^\sigma)$, that is, type (EI, EI, EIII), abbreviated as EI-I-III.

Here, we prove lemma needed and make some preparations for the theorem below.

Lemma 4.6.3. The mapping $\varphi_{\text{EI}} : Sp(4) \rightarrow (E_6)^{\lambda\gamma}$ of Theorem 3.3.1 satisfies the following equalities:

- (1) $\gamma = \varphi_{\text{EI}}(I_1)$, $\sigma = \varphi_{\text{EI}}(I_2)$.
- (2) $\gamma\varphi_{\text{EI}}(P)\gamma = \varphi_{\text{EI}}(I_1PI_1)$, $\sigma\varphi_{\text{EI}}(P)\sigma = \varphi_{\text{EI}}(I_2PI_2)$.
- (3) $\lambda(\varphi_{\text{EI}}(P)) = \varphi_{\text{EI}}(I_1PI_1)$,

where $I_1 = \text{diag}(-1, 1, 1, 1)$, $I_2 = \text{diag}(-1, -1, 1, 1)$.

Proof. The proof of (1) is omitted (see [11, Lemma 3.4.4] in detail). The equalities of (2) are the direct results of (1). As for (3), from $\lambda(\varphi_{\text{EI}}(P)) = \tau\varphi_{\text{EI}}(P)\tau$, it is easily obtained. \square

We define some element $\rho \in (E_6)^{\lambda\gamma}$ by

$$\rho = \varphi_{\text{EI}}(J_E),$$

where $J_E = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \in Sp(4)$, $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then we easily see $\rho^2 = 1$.

Consider a group $\mathcal{Z}_2 = \{1, \rho\}$. Then the group $\mathcal{Z}_2 = \{1, \rho\}$ acts on the group $Sp(2) \times Sp(2)$ by

$$\rho(A, B) = (B, A),$$

and let $(Sp(2) \times Sp(2)) \rtimes \mathcal{Z}_2$ be the semi-product of $Sp(2) \times Sp(2)$ and \mathcal{Z}_2 with this action.

Now, we determine the structure of the group $(E_6)^{\lambda\gamma} \cap (E_6)^{\lambda\gamma\sigma}$.

Theorem 4.6.4. We have that $(E_6)^{\lambda\gamma} \cap (E_6)^{\lambda\gamma\sigma} \cong (Sp(2) \times Sp(2))/\mathcal{Z}_2 \rtimes \mathcal{Z}_2$, where $\mathcal{Z}_2 = \{(E, E), (-E, -E)\}$, $\mathcal{Z}_2 = \{1, \rho\}$.

Proof. We define a mapping $\varphi_{464} : (Sp(2) \times Sp(2)) \rtimes \{1, \rho\} \rightarrow (E_6)^{\lambda\gamma} \cap (E_6)^{\lambda\gamma\sigma}$ by

$$\begin{aligned} \varphi_{464}((A, B), 1) &= \varphi_{\text{EI}}(h_1(A, B)), \\ \varphi_{464}((A, B), \rho) &= \varphi_{\text{EI}}(h_1(A, B))\rho, \end{aligned}$$

where φ_{EI} is defined in Theorem 3.3.1, and h_1 is defined as follows: $h_1 : Sp(2) \times Sp(2) \rightarrow Sp(4)$, $h_1(A, B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. From Lemma 4.6.3 (2), (3), we see that $\varphi_{464}((A, B), 1)$, $\varphi_{464}((A, B), \rho) \in (E_6)^{\lambda\gamma} \cap (E_6)^{\lambda\gamma\sigma}$. Hence φ_{464} is well-defined, and using $\rho = \varphi_{\text{EI}}(J_E)$, it is easily to verify that φ_{464} is a homomorphism.

We shall show that φ_{464} is surjection. Let $\alpha \in (E_6)^{\lambda\gamma} \cap (E_6)^{\lambda\gamma\sigma}$. Since $\alpha \in (E_6)^{\lambda\gamma} \cap (E_6)^{\lambda\gamma\sigma} \subset (E_6)^{\lambda\gamma}$, there exists $P \in Sp(4)$ such that $\alpha = \varphi_{\text{EI}}(P)$ (Theorem 3.3.1). Moreover, from $\alpha = \varphi_{\text{EI}}(P) \in (E_6)^{\lambda\gamma\sigma}$, that is, $(\lambda\gamma\sigma)\varphi_{\text{EI}}(P)(\lambda\gamma\sigma)^{-1} = \varphi_{\text{EI}}(P)$, using $\gamma\varphi_{\text{EI}}(P)\gamma = \varphi_{\text{EI}}(I_1PI_1)$, $\sigma\varphi_{\text{EI}}(P)\sigma = \varphi_{\text{EI}}(I_2PI_2)$ and $\lambda(\varphi_{\text{EI}}(P)) = \varphi_{\text{EI}}(I_1PI_1)$ (Lemma 4.6.3 (2), (3)), we have that $\varphi_{\text{EI}}(I_2PI_2) = \varphi_{\text{EI}}(P)$. Hence, it follows that

$$I_2PI_2 = P \quad \text{or} \quad I_2PI_2 = -P.$$

In the former case, we easily get the explicit form of $P \in Sp(4)$ as follows:

$$P = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad A, B \in Sp(2).$$

Hence, for $\alpha \in (E_6)^{\lambda\gamma} \cap (E_6)^{\lambda\gamma\sigma}$, we have that

$$\alpha = \varphi_{\text{EI}}\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right) = \varphi_{\text{EI}}(h_1(A, B)) = \varphi_{464}((A, B), 1).$$

In the latter case, as the former case, we can also find the explicit form of $P \in Sp(4)$ as follows:

$$P = \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}, \quad C, D \in Sp(2).$$

Hence, for $\alpha \in (E_6)^{\lambda\gamma} \cap (E_6)^{\lambda\gamma\sigma}$, we have that

$$\begin{aligned} \alpha &= \varphi_{\text{EI}} \left(\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \right) = \varphi_{\text{EI}} \left(\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \right) = \varphi_{\text{EI}} \left(\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \right) \varphi_{\text{EI}} \left(\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \right) \\ &= \varphi_{\text{EI}}(h_1(C, D))\rho = \varphi_{464}((C, D), \rho). \end{aligned}$$

Thus φ_{464} is surjection.

From $\text{Ker } \varphi_{\text{EI}} = \{E, -E\}$, we can easily obtain that $\text{Ker } \varphi_{464} = \{((E, E), 1), ((-E, -E), 1)\} \cong (\mathbf{Z}_2, 1)$.

Therefore we have the required isomorphism

$$(E_6)^{\lambda\gamma} \cap (E_6)^{\lambda\gamma\sigma} \cong (Sp(2) \times Sp(2))/\mathbf{Z}_2 \rtimes \mathbf{Z}_2.$$

□

4.7. Type EI-II-IV. In this section, we give a pair of involutive automorphisms $\lambda\tilde{\gamma}$ and $\tilde{\gamma}$.

From the result of type EI, EII, EIV in Table 2, we have the following theorem.

Theorem 4.7.1. *For $\mathbf{Z}_2 \times \mathbf{Z}_2 = \{1, \lambda\gamma\} \times \{1, \gamma\}$, the $\mathbf{Z}_2 \times \mathbf{Z}_2$ -symmetric space is of type $(E_6/(E_6)^{\lambda\gamma}, E_6/(E_6)^\gamma, E_6/(E_6)^{(\lambda\gamma)\gamma}) = (E_6/(E_6)^{\lambda\gamma}, E_6/(E_6)^\gamma, E_6/(E_6)^\lambda)$, that is, type (EI, EII, EIV), abbreviated as EI-II-IV.*

Now, we determine the structure of the group $(E_6)^{\lambda\gamma} \cap (E_6)^\gamma$.

Theorem 4.7.2. *We have that $(E_6)^{\lambda\gamma} \cap (E_6)^\gamma \cong (Sp(1) \times Sp(3))/\mathbf{Z}_2$, $\mathbf{Z}_2 = \{(1, E), (-1, -E)\}$.*

Proof. We define a mapping $\varphi_{472} : Sp(1) \times Sp(3) \rightarrow (E_6)^{\lambda\gamma} \cap (E_6)^\gamma$ by

$$\varphi_{472}(p, A) = \varphi_{\text{EI}}(h_2(p, A)),$$

where h_2 is defined by $h_2 : Sp(1) \times Sp(3) \rightarrow Sp(4)$, $h_2(p, A) = \begin{pmatrix} p & 0 \\ 0 & A \end{pmatrix}$. Since the mapping φ_{472} is the restriction of the mapping φ_{EI} , it is easily to verify that φ_{472} is well-defined and a homomorphism.

We shall show that φ_{472} is surjection. Let $\alpha \in (E_6)^{\lambda\gamma} \cap (E_6)^\gamma$. Since $\alpha \in (E_6)^{\lambda\gamma} \cap (E_6)^\gamma \subset (E_6)^{\lambda\gamma}$, there exists $P \in Sp(4)$ such that $\alpha = \varphi_{\text{EI}}(P)$ (Theorem 3.3.1). Moreover, from $\alpha = \varphi_{\text{EI}}(P) \in (E_6)^\gamma$, that is, $\gamma\varphi_{\text{EI}}(P)\gamma = \varphi_{\text{EI}}(P)$, using $\gamma\varphi_{\text{EI}}(P)\gamma = \varphi_{\text{EI}}(I_1 P I_1)$ (Lemma 4.6.3 (2)) we have that $\varphi_{\text{EI}}(I_1 P I_1) = \varphi_{\text{EI}}(P)$. Hence it follows that

$$I_1 P I_1 = P \quad \text{or} \quad I_1 P I_1 = -P.$$

In the former case, we easily get the explicit form of $P \in Sp(4)$ as follows:

$$P = \begin{pmatrix} p & 0 \\ 0 & A \end{pmatrix}, \quad p \in Sp(1), A \in Sp(3).$$

Hence for $\alpha \in (E_6)^{\lambda\gamma} \cap (E_6)^{\lambda\gamma\sigma}$, we have that

$$\alpha = \varphi_{\text{EI}} \left(\begin{pmatrix} p & 0 \\ 0 & A \end{pmatrix} \right) = \varphi_{\text{EI}}(h_2(p, A)) = \varphi_{472}((p, A)).$$

In the latter case, as the former case, we can also find the explicit form of $P \in Sp(4)$ as follows:

$$P = \begin{pmatrix} 0 & b & c & d \\ l & 0 & 0 & 0 \\ m & 0 & 0 & 0 \\ n & 0 & 0 & 0 \end{pmatrix}, \quad b, c, d, l, m, n \in \mathbf{H}.$$

This is contrary to the condition $P \in Sp(4)$ because of $b = c = d = 0$. Hence this case is impossible. Thus φ_{472} is surjection.

From $\text{Ker } \varphi_{E_1} = \{(1, E), (-1, -E)\}$, we can easily obtain that $\text{Ker } \varphi_{472} = \{(1, E), (-1, -E)\} \cong \mathbb{Z}_2$.

Therefore we have the required isomorphism

$$(E_6)^{\lambda\gamma} \cap (E_6)^\gamma \cong (Sp(1) \times Sp(3))/\mathbb{Z}_2.$$

□

4.8. Type EII-II-II. In this section, we give a pair of involutive inner automorphisms $\tilde{\gamma}$ and $\tilde{\gamma}_H$.

Again, let the C -linear transformation γ_H of \mathfrak{J}^C . As γ_c in section 4.5, γ_H induces involutive inner automorphism $\tilde{\gamma}_H$ of E_6 : $\tilde{\gamma}_H(\alpha) = \gamma_H \alpha \gamma_H, \alpha \in E_6$.

Using the inclusion $F_4 \subset E_6$, the \mathbf{R} -linear transformations δ_1, δ_2 defined in the proof of Lemma 4.1.1 are naturally extended to the C -linear transformations of \mathfrak{J}^C . Obviously, we have $\delta_1, \delta_2 \in E_6, \delta_1^2 = \delta_2^2 = 1$. Hence, as in G_2 and F_4 , since we easily see that $\delta_1 \gamma = \gamma_H \delta_1, \delta_2 \gamma = (\gamma \gamma_H) \delta_2$, that is, $\gamma \sim \gamma_H, \gamma \sim \gamma \gamma_H$ in E_6 , we have the following proposition.

Proposition 4.8.1. *The group $(E_6)^\gamma$ is isomorphic to both of the groups $(E_6)^{\gamma_H}$ and $(E_6)^{\gamma \gamma_H}$: $(E_6)^\gamma \cong (E_6)^{\gamma_H} \cong (E_6)^{\gamma \gamma_H}$.*

From the result of type EII in Table 2 and Proposition 4.8.1, we have the following theorem.

Theorem 4.8.2. *For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \gamma\} \times \{1, \gamma_H\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(E_6/(E_6)^\gamma, E_6/(E_6)^{\gamma_H}, E_6/(E_6)^{\gamma \gamma_H})$, that is, type (EII, EII, EII), abbreviated as EII-II-II.*

Here, we prove proposition needed and make some preparations for the theorem below.

We define spaces $G_{3,3}$ and $G_{3,3}^-$ by $\{U \in SU(6) \mid IUI = U\}$ and $\{U \in SU(6) \mid IUI = -U\}$:

$$G_{3,3} = \{U \in SU(6) \mid IUI = U\}, \quad G_{3,3}^- = \{U \in SU(6) \mid IUI = -U\},$$

respectively, where $I = \text{diag}(1, -1, 1, -1, 1, -1)$.

$$\text{Moreover, we consider some element } M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in SO(6) \subset SU(6)$$

such that $IM_1 = M_1 I_3$, where $I_3 = \text{diag}(1, 1, 1, -1, -1, -1)$.

Proposition 4.8.3. *We have the following isomorphisms:*

- (1) $G_{3,3} \cong S(U(3) \times U(3))$ (as a group), where $S(U(3) \times U(3)) = \{U \in SU(6) \mid I_3 U I_3 = U\}$.
- (2) $G_{3,3}^- \cong S(U(3) \times U(3))^-$ (as a set), where $S(U(3) \times U(3))^- = \{U \in SU(6) \mid I_3 U I_3 = -U\}$.
- (3) $S(U(3) \times U(3)) \cong (U(1) \times SU(3) \times SU(3))/\mathbb{Z}_3$, $\mathbb{Z}_3 = \{(1, E, E), (\omega_1, \omega_1^2, \omega_1 E), (\omega_1^2, \omega_1, \omega_1^2 E)\}$, where $\omega \in \mathbb{C}, \omega^3 = 1, \omega \neq 1$.

Proof. (1) We define a homomorphism $k_1 : S(U(3) \times U(3)) \rightarrow G_{3,3}$ by

$$k_1(U) = M_1 U M_1^{-1}.$$

Then it is easily to verify that k_1 is isomorphism as a Lie group.

(2) We define a mapping $k_1^- : S(U(3) \times U(3))^- \rightarrow G_{3,3}^-$ by

$$k_1^-(U^-) = M_1 U^- M_1^{-1}.$$

Then it is easily to verify that k_1^- is isomorphism as a set.

(3) We define a homomorphism $h_3 : U(1) \times SU(3) \times SU(3) \rightarrow S(U(3) \times U(3))$ by

$$h_3(\theta, A, B) = \begin{pmatrix} \theta A & 0 \\ 0 & \theta^{-1} B \end{pmatrix}.$$

Then we can easily show that the mapping h_3 induces the required isomorphism. \square

We define some element $\rho_1 \in (E_6)^\gamma$ by

$$\rho_1 = \varphi_{E_2}(e_2, \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}), \quad E = \text{diag}(1, 1, 1) \in M(3, \mathbf{R}),$$

where φ_{E_2} is defined in Theorem 3.3.2. Hereafter, we denote the matrix $\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ by E_J .

Then we remark that E_J commutes with M_1 : $E_J M_1 = M_1 E_J$.

Consider a group $\mathcal{Z}_2 = \{1, \rho_1\}$. Then the group $\mathcal{Z}_2 = \{1, \rho_1\}$ acts on the group $U(1) \times (U(1) \times SU(3) \times SU(3))$ by

$$\rho_1(a, (\theta, A, B)) = (\bar{a}, (\theta^{-1}, B, A))$$

and let $U(1) \times (U(1) \times SU(3) \times SU(3)) \rtimes \mathcal{Z}_2$ be the semi-direct product of $U(1) \times (U(1) \times SU(3) \times SU(3))$ and \mathcal{Z}_2 with this action.

Now, we determine the structure of the group $(E_6)^\gamma \cap (E_6)^{\gamma_H}$.

Theorem 4.8.4. *We have that $(E_6)^\gamma \cap (E_6)^{\gamma_H} \cong (U(1) \times (U(1) \times SU(3) \times SU(3))) / (\mathcal{Z}_2 \times \mathcal{Z}_3) \rtimes \mathcal{Z}_2$, $\mathcal{Z}_2 = \{(1, 1, E, E), (-1, -1, -E, -E)\}$, $\mathcal{Z}_3 = \{(1, 1, E, E), (1, \omega, \omega^2, \omega E), (1, \omega^2, \omega, \omega^2 E)\}$, $\mathcal{Z}_2 = \{1, \rho_1\}$, where $\omega \in \mathbf{C}$, $\omega^3 = 1$, $\omega \neq 1$.*

Proof. We define a mapping $\varphi_{484} : (U(1) \times (U(1) \times SU(3) \times SU(3))) \rtimes \{1, \rho_1\} \rightarrow (E_6)^\gamma \cap (E_6)^{\gamma_H}$ by

$$\begin{aligned} \varphi_{484}((a, (\theta, A, B)), 1) &= \varphi_{E_2}(a, k_1 h_3(\theta, A, B)), \\ \varphi_{484}((a, (\theta, A, B)), \rho_1) &= \varphi_{E_2}(a, k_1 h_3(\theta, A, B)) \rho_1, \end{aligned}$$

where k_1, h_3 are defined in Proposition 4.8.3 (1), (3), respectively. From Lemma 4.5.4 (2), we have $\varphi_{484}((a, (\theta, A, B)), 1), \varphi_{484}((a, (\theta, A, B)), \rho_1) \in (E_6)^\gamma \cap (E_6)^{\gamma_H}$. Hence φ_{484} is well-defined. Using $\rho_1 = \varphi_{E_2}(e_2, \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix})$, we can confirm that φ_{484} is a homomorphism. Indeed, we show that the case of $\varphi_{484}((a, (\theta, A, B)), \rho_1) \varphi_{484}((b, (\nu, C, D)), 1) = \varphi_{484}(((a, (\theta, A, B)), \rho_1)((b, (\nu, C, D)), 1))$ as example. For the left hand side of this equality, we have that

$$\begin{aligned} &\varphi_{484}((a, (\theta, A, B)), \rho_1) \varphi_{484}((b, (\nu, C, D)), 1) \\ &= (\varphi_{E_2}(a, k_1 h_3(\theta, A, B)) \rho_1) (\varphi_{E_2}(b, k_1 h_3(\nu, C, D))) \\ &= \varphi_{E_2}(a, M_1 h_3(\theta, A, B) M_1) \varphi_{E_2}(e_2, E_J) (\varphi_{E_2}(b, M_1 h_3(\nu, C, D) M_1)) \\ &= \varphi_{E_2}((a e_2) b, M_1 h_3(\theta, A, B) M_1 E_J M_1 h_3(\nu, C, D) M_1) \\ &= \varphi_{E_2}((\bar{a} \bar{b}) e_2, M_1 h_3(\theta, A, B) h_3(\nu^{-1}, D, C) M_1 E_J) \\ &= \varphi_{E_2}(\bar{a} \bar{b}, k_1 h_3(\theta \nu^{-1}, AD, BC)) \rho_1. \end{aligned}$$

On the other hand, for the right hand side of same one, we have that

$$\begin{aligned}
 & \varphi_{484}(((a, (\theta, A, B)), \rho_1)((b, (\nu, C, D)), 1)) \\
 &= \varphi_{484}((a, (\theta, A, B))\rho_1(b, (\nu, C, D)), \rho_1) \\
 &= \varphi_{484}(a\bar{b}, (\theta, A, B)(\nu^{-1}, D, C), \rho_1) \\
 &= \varphi_{484}(a\bar{b}, (\theta\nu^{-1}, AD, BC), \rho_1) \\
 &= \varphi_{\mathbb{E}_2}(a\bar{b}, k_1h_3(\theta\nu^{-1}, AD, BC))\rho_1,
 \end{aligned}$$

that is, $\varphi_{484}((a, (\theta, A, B)), \rho_1)\varphi_{484}((b, (\nu, C, D)), 1) = \varphi_{484}(((a, (\theta, A, B)), \rho_1)((b, (\nu, C, D)), 1))$. Similarly, the other cases are shown.

We shall show that φ_{484} is surjection. Let $\alpha \in (E_6)^\gamma \cap (E_6)^{\gamma_H}$. Since $\alpha \in (E_6)^\gamma \cap (E_6)^{\gamma_H} \subset (E_6)^\gamma$, there exist $p \in Sp(1)$ and $U \in SU(6)$ such that $\alpha = \varphi_{\mathbb{E}_2}(p, U)$ (Theorem 3.3.2). Moreover, from $\alpha = \varphi_{\mathbb{E}_2}(p, U) \in (E_6)^{\gamma_H}$, that is, $\gamma_H\varphi_{\mathbb{E}_2}(p, U)\gamma_H = \varphi_{\mathbb{E}_2}(p, U)$, using $\gamma_H\varphi_{\mathbb{E}_2}(p, U)\gamma_H = \varphi_{\mathbb{E}_2}(\gamma_H p, IUI)$ (Lemma 4.5.4 (2)), we have $\varphi_{\mathbb{E}_2}(\gamma_H p, IUI) = \varphi_{\mathbb{E}_2}(p, U)$. Hence it follows that

$$\begin{cases} \gamma_H p = p \\ IUI = U \end{cases} \quad \text{or} \quad \begin{cases} \gamma_H p = -p \\ IUI = -U. \end{cases}$$

In the former case, we see that $p \in U(1)$ and $U \in G_{3,3}$. Since there exist $\theta \in U(1)$ and $A, B \in SU(3)$ such that $U = k_1h_3(\theta, A, B)$ for $U \in G_{3,3}$ (Proposition 4.8.3 (1), (3)), we have that $\alpha = \varphi_{\mathbb{E}_2}(a, k_1h_3(\theta, A, B)) = \varphi_{484}(a, (\theta, A, B), 1)$. In the latter case, first we get the explicit form of $p \in Sp(1)$ as follows:

$$p = p_2e_2 + p_3e_3 = be_2 \ (b = p_2 + p_3e_1 \in U(1)),$$

moreover since $U \in G_{3,3}^-$, there exists $U^- \in S(U(3) \times U(3))^-$ such that $U = k_1^-(U^-)$ (Proposition 4.8.3 (2)), that is, there exist $C, D \in U(3)$ such that $U = k_1^-\left(\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}\right), (\det C)(\det D) = -1$. Hence, from $\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & -D \end{pmatrix} E_J$, we have that

$$\begin{aligned}
 U &= k_1^-\left(\begin{pmatrix} C & 0 \\ 0 & -D \end{pmatrix}\right)\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} = M_1\left(\begin{pmatrix} C & 0 \\ 0 & -D \end{pmatrix} E_J\right)M_1^{-1} \\
 &= (M_1\begin{pmatrix} C & 0 \\ 0 & -D \end{pmatrix}M_1^{-1})(M_1E_JM_1^{-1}) \cdots (*).
 \end{aligned}$$

Here, since $\begin{pmatrix} C & 0 \\ 0 & -D \end{pmatrix} \in S(U(3) \times U(3))$ and $M_1E_J = E_JM_1$, we modify (*) above as follows: $(*) = k_1\left(\begin{pmatrix} C & 0 \\ 0 & -D \end{pmatrix}\right)E_J$. Hence, since there exist $\nu \in U(1)$ and $L, N \in SU(3)$ such that $U = k_1h_3(\nu, L, N)E_J$ (Proposition 4.8.3 (3)), we have that

$$\begin{aligned}
 \alpha &= \varphi_{\mathbb{E}_2}(be_2, k_1h_3(\nu, L, N)E_J) = \varphi_{\mathbb{E}_2}(b, k_1h_3(\nu, L, N))\varphi_{\mathbb{E}_2}(e_2, E_J) \\
 &= \varphi_{\mathbb{E}_2}(b, k_1h_3(\nu, L, N))\rho_1 = \varphi_{484}((b, (\nu, L, N)), \rho_1).
 \end{aligned}$$

Thus φ_{484} is surjection.

From $\text{Ker } \varphi_{\mathbb{E}_2} = \{(1, E), (-1, -E)\}$ and $\text{Ker } h_3 = \{(1, E, E), (\omega, \omega^2, \omega E), (\omega^2, \omega, \omega^2 E)\}$, we can easily obtain that

$$\begin{aligned}
 \text{Ker } \varphi_{484} &= \{(1, (1, E, E), 1), (-1, (-1, -E, -E), 1)\} \\
 &\quad \times \{(1, (1, E, E), 1), (1, (\omega, \omega^2, \omega E), 1), (1, (\omega^2, \omega, \omega^2 E), 1)\} \\
 &\cong (\mathbb{Z}_2 \times \mathbb{Z}_3, 1),
 \end{aligned}$$

where $\omega \in C, \omega^3 = 1, \omega \neq 1$.

Therefore we have the required isomorphism

$$(E_6)^\gamma \cap (E_6)^{\gamma_H} \cong (U(1) \times (U(1) \times SU(3) \times SU(3)))/(\mathbf{Z}_2 \times \mathbf{Z}_3) \rtimes \mathbf{Z}_2.$$

□

4.9. Type EII-II-III. In this section, we use a pair of involutive inner automorphisms $\tilde{\gamma}$ and $\gamma\tilde{\sigma}$.

Since $\gamma \sim \gamma\sigma$ in E_6 as mentioned in Section 4.6, we have the following proposition which is the direct result of this.

Proposition 4.9.1. *The group $(E_6)^\gamma$ is isomorphic to the group $(E_6)^{\gamma\sigma}$: $(E_6)^\gamma \cong (E_6)^{\gamma\sigma}$.*

From the result of types EII, EIII in Table 2 and Proposition 4.9.1, we have the following theorem.

Theorem 4.9.2. *For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \gamma\} \times \{1, \gamma\sigma\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(E_6/(E_6)^\gamma, E_6/(E_6)^{\sigma\gamma}, E_6/(E_6)^{(\gamma)(\gamma\sigma)}) = (E_6/(E_6)^\gamma, E_6/(E_6)^\gamma, E_6/(E_6)^\sigma)$, that is, type (EII, EII, EIII), abbreviated as EII-II-III.*

Here, we prove Proposition needed in theorem below.

Proposition 4.9.3. *We have the following isomorphism: $(U(1) \times Sp(1) \times SU(4))/\mathbf{Z}_4 \cong S(U(2) \times U(4))$, $\mathbf{Z}_4 = \{(1, 1, E), (-1, -1, -E), (i, -1, iE), (-i, -1, -iE)\}$.*

Proof. We define a mapping

$$f : U(1) \times Sp(1) \times SU(4) \xrightarrow{k} U(1) \times SU(2) \times SU(4) \xrightarrow{h_4} S(U(2) \times U(4))$$

by

$$f(\theta, q, A) = h_4(\theta, k(q), A),$$

where a isomorphism k is defined by $k : Sp(1) \rightarrow SU(2)$, $k(a + be_2) = \begin{pmatrix} a' & b' \\ -\tau b' & \tau a' \end{pmatrix}$ and a

homomorphism $h_4 : U(1) \times SU(2) \times SU(4) \rightarrow S(U(2) \times U(4))$ by $h_4(\theta, C, A) = \begin{pmatrix} \theta^2 C & 0 \\ 0 & \theta^{-1} A \end{pmatrix}$.

(Remark. For $a = a_1 + a_2 e_1 \in C$, we express a' as the components replacing e_1 by i , that is, $a' = a_1 + a_2 i$. It is similar to a' as for b' , so is the components of $SU(4)$.) We can show easily that the homomorphism f induces the required isomorphism. □

Now, we determine the structure of the group $(E_6)^\gamma \cap (E_6)^{\gamma\sigma}$.

Theorem 4.9.4. *We have that $(E_6)^\gamma \cap (E_6)^{\gamma\sigma} \cong (Sp(1) \times (U(1) \times Sp(1) \times SU(4)))/(\mathbf{Z}_2 \times \mathbf{Z}_4)$, $\mathbf{Z}_2 = \{(1, (1, 1, E)), (-1, (-1, -1, -E))\}$, $\mathbf{Z}_4 = \{(1, (1, 1, E)), (1, (-1, -1, -E)), (1, (i, -1, iE)), (1, (-i, -1, -iE))\}$.*

Proof. We define a mapping $\varphi_{494} : Sp(1) \times (U(1) \times Sp(1) \times SU(4)) \rightarrow (E_6)^\gamma \cap (E_6)^{\gamma\sigma}$ by

$$\varphi_{494}(p, (\theta, q, A)) = \varphi_{E_2}(p, f(\theta, q, A)),$$

where f is defined in Proposition 4.9.3. Since the mapping φ_{494} is the restriction of the mapping φ_{E_2} , it is easily to verify that φ_{494} is well-defined and a homomorphism.

We shall show that φ_{494} is surjection. Let $\alpha \in (E_6)^\gamma \cap (E_6)^{\gamma\sigma}$. Since $(E_6)^\gamma \cap (E_6)^{\gamma\sigma} \subset (E_6)^\gamma$, there exist $p \in Sp(1)$ and $U \in SU(6)$ such that $\alpha = \varphi_{E_2}(p, U)$ (Theorem 3.3.2). Moreover, from $\alpha = \varphi_{E_2}(p, U) \in (E_6)^{\gamma\sigma}$, that is, $(\gamma\sigma)\varphi_{E_2}(p, U)(\sigma\gamma) = \varphi_{E_2}(p, U)$, using

$\gamma\varphi_{\mathbb{E}_2}(p, U)\gamma = \varphi_{\mathbb{E}_2}(p, U)$ and $\sigma\varphi_{\mathbb{E}_2}(p, U)\sigma = \varphi_{\mathbb{E}_2}(p, I_2UI_2)$ (Lemma 4.5.4 (2)), we easily see $\varphi_{\mathbb{E}_2}(p, I_2UI_2) = \varphi_{\mathbb{E}_2}(p, U)$. Hence it follows that

$$\begin{cases} p = p \\ I_2UI_2 = U \end{cases} \quad \text{or} \quad \begin{cases} p = -p \\ I_2UI_2 = -U. \end{cases}$$

In the former case, we see that $p \in Sp(1)$ and $U \in S(U(2) \times U(4))$. Moreover for $U \in S(U(2) \times U(4))$, there exist $\theta \in U(1)$, $q \in Sp(1)$ and $A \in SU(4)$ such that $U = f(\theta, q, A)$ (Proposition 4.9.3). Hence we have that $\alpha = \varphi_{\mathbb{E}_2}(p, f(\theta, q, A)) = \varphi_{494}(p, (\theta, q, A))$. In the latter case, this is contrary to the condition $p \in Sp(1)$ because of $p = 0$. Hence this case is impossible. Thus φ_{494} is surjection.

From $\text{Ker } \varphi_{\mathbb{E}_2} = \{(1, E), (-1, -E)\}$ and $\text{Ker } f = \{(1, 1, E), (-1, -1, -E), (i, -1, iE), (-i, -1, -iE)\}$, we have easily obtain that

$$\begin{aligned} \text{Ker } \varphi_{494} &= \{(1, (1, 1, E)), (-1, (-1, -1, -E))\} \times \{(1, (1, 1, E)), (1, (-1, -1, -E)), \\ &\quad (1, (i, -1, iE)), (1, (-i, -1, -iE))\} \cong \mathbb{Z}_2 \times \mathbb{Z}_4. \end{aligned}$$

Therefore we have the required isomorphism

$$(E_6)^\gamma \cap (E_6)^{\gamma\sigma} \cong (Sp(1) \times (U(1) \times Sp(1) \times SU(4)))/(\mathbb{Z}_2 \times \mathbb{Z}_4).$$

□

4.10. Type EII-III-III. In this section, we give a pair of involutive inner automorphisms $\tilde{\gamma}$ and $\gamma_{\mathbb{H}}\tilde{\rho}_2$, where $\gamma_{\mathbb{H}}\tilde{\rho}_2$ is induced by a C -linear transformation $\gamma_{\mathbb{H}}\rho_2$ of \mathfrak{J}^C : $\gamma_{\mathbb{H}}\tilde{\rho}_2(\alpha) = (\gamma_{\mathbb{H}}\rho_2)\alpha(\rho_2\gamma_{\mathbb{H}})$, $\alpha \in E_6$, and a C -linear transformation ρ_2 of \mathfrak{J}^C is defined below.

We define some element $\rho_2 \in (E_6)^\gamma$ by

$$\rho_2 = \varphi_{\mathbb{E}_2}(1, L_2),$$

where $L_2 = \text{diag}(1, 1, -1, 1, -1, 1) \in SO(6) \subset SU(6)$, and the explicit form of ρ_2 as action to \mathfrak{J}^C is given by

$$\rho_2 X = \begin{pmatrix} \xi_1 & -ix_3e_1 & i\bar{x}_2e_1 \\ ie_1\bar{x}_3 & -\xi_2 & e_1x_1e_1 \\ ie_1x_2 & e_1\bar{x}_1e_1 & -\xi_3 \end{pmatrix}, X \in \mathfrak{J}^C.$$

Then we have that $\rho_2^2 = 1$, $\delta_1\rho_2 = \rho_2\delta_1$, $\delta_2\rho_2 = \rho_2\delta_2$, where δ_1, δ_2 are the same ones used in Section 4.8.

$$\text{Now, for } D_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in SO(6) \subset SU(6), \text{ we consider some element}$$

$\varphi_{\mathbb{E}_2}(1, D_8) \in (E_6)^\gamma$, and we denote $\varphi_{\mathbb{E}_2}(1, D_8)$ by δ_8 : $\delta_8 = \varphi_{\mathbb{E}_2}(1, D_8)$. Then since $\gamma\sigma = \varphi_{\mathbb{E}_2}(1, I_2)$, we have $\delta_8(\gamma\sigma) = \rho_2\delta_8$. Hence, together with $\gamma\rho_2 = \rho_2\gamma$, we have $\delta_8^{-1}(\gamma\rho_2) = \sigma\delta_8^{-1}$.

Lemma 4.10.1. In E_6 , σ is conjugate to both of $\gamma_{\mathbb{H}}\rho_2$ and $\gamma\gamma_{\mathbb{H}}\rho_2$: $\sigma \sim \gamma_{\mathbb{H}}\rho_2, \sigma \sim \gamma\gamma_{\mathbb{H}}\rho_2$.

Proof. Using $\delta_8^{-1}(\gamma\rho_2) = \sigma\delta_8^{-1}$, $\delta_k\rho_2 = \rho_2\delta_k, k = 1, 2$, we have that

$$\begin{aligned} (\delta_8^{-1}\delta_1)(\gamma_{\mathbb{H}}\rho_2) &= \delta_8^{-1}(\delta_1\gamma_{\mathbb{H}})\rho_2 = \delta_8^{-1}(\gamma\delta_1)\rho_2 \\ &= \delta_8^{-1}\gamma(\delta_1\rho_2) = \delta_8^{-1}\gamma(\rho_2\delta_1) = (\delta_8^{-1}(\gamma\rho_2))\delta_1 \\ &= (\sigma\delta_8^{-1})\delta_1 = \sigma(\delta_8^{-1}\delta_1), \end{aligned}$$

that is, $\sigma \sim \gamma_H \rho_2$. Moreover, we have $\sigma \sim \gamma \gamma_H \rho_2$ in the same way as the former case. Indeed,

$$\begin{aligned} (\delta_8^{-1} \delta_2)(\gamma \gamma_H \rho_2) &= \delta_8^{-1} (\delta_2 \gamma \gamma_H) \rho_2 = \delta_8^{-1} (\gamma \delta_2) \rho_2 \\ &= \delta_8^{-1} \gamma (\delta_2 \rho_2) = \delta_8^{-1} \gamma (\rho_2 \delta_2) = (\delta_8^{-1} (\gamma \rho_2)) \delta_2 \\ &= (\sigma \delta_8^{-1}) \delta_2 = \sigma (\delta_8^{-1} \delta_2), \end{aligned}$$

that is, $\sigma \sim \gamma \gamma_H \rho_2$. □

We have the following proposition which is the direct result of Lemma 4.10.1.

Proposition 4.10.2. *The group $(E_6)^\sigma$ is isomorphic to both of the groups $(E_6)^{\gamma_H \rho_2}$ and $(E_6)^{\gamma \gamma_H \rho_2}$: $(E_6)^\sigma \cong (E_6)^{\gamma_H \rho_2} \cong (E_6)^{\gamma \gamma_H \rho_2}$.*

From the result of types EII, EIII in Table 2 and Proposition 4.10.2, we have the following theorem.

Theorem 4.10.3. *For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \gamma\} \times \{1, \gamma_H \rho_2\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(E_6/(E_6)^\gamma, E_6/(E_6)^{\gamma_H \rho_2}, E_6/(E_6)^{\gamma(\gamma_H \rho_2)}) = (E_6/(E_6)^\gamma, E_6/(E_6)^\sigma, E_6/(E_6)^\sigma)$, that is, type (EII, EIII, EIII), abbreviated as EII-III-III.*

Here, we prove proposition needed in theorem below.

Proposition 4.10.4. *We have the following isomorphism: $(U(1) \times SU(5))/\mathbf{Z}_5 \cong S(U(1) \times U(5))$, $\mathbf{Z}_5 = \{(1, \nu^k E) \mid \nu \in C, \nu^5 = 1, k = 0, 1, 2, 3, 4\}$.*

Proof. We define a mapping $h_5 : U(1) \times SU(5) \rightarrow S(U(1) \times U(5))$ by

$$h_5(\theta, A) = \begin{pmatrix} \theta^5 & 0 \\ 0 & \theta^{-1} A \end{pmatrix}.$$

Then we can easily show that h_5 induces the required isomorphism. □

Now, we determine the structure of the group $(E_6)^\gamma \cap (E_6)^{\gamma_H \rho_2}$.

Theorem 4.10.5. *We have that $(E_6)^\gamma \cap (E_6)^{\gamma_H \rho_2} \cong (U(1) \times U(1) \times SU(5))/(\mathbf{Z}_2 \times \mathbf{Z}_5)$, $\mathbf{Z}_2 = \{(1, 1, E), (-1, -1, -E)\}$, $\mathbf{Z}_5 = \{(1, \nu^k, \nu^k E)\}$, $k = 0, 1, 2, 3, 4$.*

Proof. We define a mapping $\varphi_{4105} : U(1) \times U(1) \times SU(5) \rightarrow (E_6)^\gamma \cap (E_6)^{\gamma_H \rho_2}$ by

$$\varphi_{4105}(a, \theta, A) = \varphi_{e_2}(a, h_5(\theta, A)),$$

where h_5 is defined in Proposition 4.10.4. Since the mapping φ_{4105} is the restriction of the mapping φ_{e_2} , it is easily to verify that φ_{4105} is well-defined and a homomorphism.

We shall show that φ_{4105} is surjection. Let $\alpha \in (E_6)^\gamma \cap (E_6)^{\gamma_H \rho_2}$. Since $(E_6)^\gamma \cap (E_6)^{\gamma_H \rho_2} \subset (E_6)^\gamma$, there exist $p \in Sp(1)$ and $U \in SU(6)$ such that $\alpha = \varphi_{e_2}(p, U)$ (Theorem 3.3.2). Moreover, from $\alpha = \varphi_{e_2}(p, U) \in (E_6)^{\gamma_H \rho_2}$, that is, $(\gamma_H \rho_2) \varphi_{e_2}(p, U) (\rho_2 \gamma_H) = \varphi_{e_2}(p, U)$, using $\gamma_H \varphi_{e_2}(p, U) \gamma_H = \varphi_{e_2}(\gamma_H p, IUI)$ (Lemma 4.5.4 (2)) and $\rho_2 = \varphi_{e_2}(1, L_2)$, we have that $\varphi_{e_2}(\gamma_H p, (IL_2)U(L_2I)) = \varphi_{e_2}(p, U)$. Hence it follows that

$$\begin{cases} \gamma_H p = p \\ (IL_2)U(L_2I) = U \end{cases} \quad \text{or} \quad \begin{cases} \gamma_H p = -p \\ (IL_2)U(L_2I) = -U. \end{cases}$$

In the former case, we see that $p \in U(1)$, moreover since $IL_2 = L_2I = \text{diag}(1, -1, -1, -1, -1, -1)$, we get the explicit form of $U \in SU(6)$ as follows:

$$U = \begin{pmatrix} \zeta & 0 \\ 0 & B \end{pmatrix}, \zeta \in U(1), B \in U(5), \det U = 1,$$

that is, $U \in S(U(1) \times U(5))$. Hence, since there exist $\theta \in U(1)$ and $A \in SU(5)$ such that $U = h_5(\theta, A)$ (Proposition 4.10.4), we have $\alpha = \varphi_{\mathbb{E}_2}(a, h_5(\theta, A)) = \varphi_{4105}(a, \theta, A)$. In the latter case, as the former case, we can also find the explicit form of $U \in SU(6)$ as follows:

$$U = \begin{pmatrix} 0 & \mathbf{x} \\ \mathbf{y} & 0 \end{pmatrix}, \quad \mathbf{x}, \mathbf{y} \in \mathbb{C}^5,$$

where $C = \{x_1 + x_2 i \mid x_k \in \mathbb{R}, k = 1, 2\}$. However, this case is impossible because of $\det U = 0$ for $U \in SU(6)$. Thus φ_{4105} is surjection.

From $\text{Ker } \varphi_{\mathbb{E}_2} = \{(1, E), (-1, -E)\}$ and $\text{Ker } h_5 = \{(\nu^k, \nu^k E) \mid k = 0, 1, 2, 3, 4\}$, we can easily obtain that $\text{Ker } \varphi_{4105} = \{(1, 1, E), (-1, -1, -E)\} \times \{(\nu^k, \nu^k E) \mid k = 0, 1, 2, 3, 4\} \cong \mathbb{Z}_2 \times \mathbb{Z}_5$.

Therefore we have the required isomorphism

$$(E_6)^\gamma \cap (E_6)^{\gamma_{H^{P_2}}} \cong (U(1) \times U(1) \times SU(5))/(\mathbb{Z}_2 \times \mathbb{Z}_5).$$

□

4.11. Type EIII-III-III. In this section, we give a pair of involutive inner automorphisms $\tilde{\sigma}$ and $\tilde{\sigma}'$.

Let the \mathbb{C} -linear transformation σ' of \mathfrak{J}^C be the complexification of $\sigma' \in F_4$, so σ' induces involutive inner automorphism $\tilde{\sigma}'$ of E_6 : $\tilde{\sigma}'(\alpha) = \sigma' \alpha \sigma', \alpha \in E_6$.

Using the inclusion $F_4 \subset E_6$, the \mathbb{R} -linear transformations δ_6, δ_7 defined in the proof of Lemma 4.4.1 are naturally extended to the \mathbb{C} -linear transformations of \mathfrak{J}^C . Then we have $\delta_6, \delta_7 \in E_6, \delta_6^2 = \delta_7^2 = 1$. Hence, as in F_4 , since we easily see that $\delta_6 \sigma = \sigma' \delta_6, \delta_7 \sigma = (\sigma \sigma') \delta_7$, that is, $\sigma \sim \sigma', \sigma \sim \sigma \sigma'$ in E_6 , we have the following proposition.

Proposition 4.11.1. *The group $(E_6)^\sigma$ is isomorphic to both of the groups $(E_6)^{\sigma'}$ and $(E_6)^{\sigma \sigma'}$: $(E_6)^\sigma \cong (E_6)^{\sigma'} \cong (E_6)^{\sigma \sigma'}$.*

From the result of Type EIII in Table 2 and Proposition 4.11.1, we have the following theorem.

Theorem 4.11.2. *For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \sigma\} \times \{1, \sigma'\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(E_6/(E_6)^\sigma, E_6/(E_6)^{\sigma'}, E_6/(E_6)^{\sigma \sigma'}) = (E_6/(E_6)^\sigma, E_6/(E_6)^\sigma, E_6/(E_6)^\sigma)$, that is, type (EIII, EIII, EIII), abbreviated as EIII-III-III.*

Here, we prove lemmas needed and make some preparations for Proposition 4.11.5 below.

First, we investigate the subgroup $((E_6)_{E_1})^{\sigma'}$ of E_6 defined by

$$((E_6)_{E_1})^{\sigma'} = \{\alpha \in E_6 \mid \alpha E_1 = E_1, \sigma' \alpha \sigma' = \alpha\},$$

where the group $(E_6)_{E_1}$ is isomorphic to $Spin(10)$ as the double covering group of $SO(10)$ (As for the group $(E_6)_{E_1} \cong Spin(10)$, see [10, Theorem 3.10.4]).

Lemma 4.11.3. *The Lie algebra $((\mathfrak{e}_6)_{E_1})^{\sigma'}$ of the group $((E_6)_{E_1})^{\sigma'}$ are given by*

$$((\mathfrak{e}_6)_{E_1})^{\sigma'} = \{\delta + i\tilde{T} \in \mathfrak{e}_6 \mid \delta \in ((\mathfrak{f}_4)_{E_1})^{\sigma'}, T \in \mathfrak{J}, \text{tr}(T) = 0, \sigma' T = T, T \circ E_1 = 0\},$$

where $((\mathfrak{f}_4)_{E_1})^{\sigma'} = (\mathfrak{f}_4)^\sigma \cap (\mathfrak{f}_4)^{\sigma'} \cong \mathfrak{so}(8)$ (Section 4.4).

In particular, we have

$$\dim(((\mathfrak{e}_6)_{E_1})^{\sigma'}) = 28 + 1 = 29.$$

Proof. Since any element ϕ of the Lie algebra \mathfrak{e}_6 of the group E_6 is uniquely as $\phi = \delta + i\tilde{T}, \delta \in \mathfrak{f}_4, T \in \mathfrak{J}_0 = \{T \in \mathfrak{J} \mid \text{tr}(T) = 0\}$, using $\sigma' \phi \sigma' = \sigma' \delta \sigma' + i(\sigma' \tilde{T})$, we can easily prove this lemma (As for $((\mathfrak{f}_4)_{E_1})^{\sigma'} \cong \mathfrak{so}(8)$, see Theorem 4.4.5 and [10, Theorem 2.9.1]). □

Lemma 4.11.4. For $\theta, \nu \in U(1) = \{\theta \in C \mid (\tau\theta)\theta = 1\}$, we define C -linear transformations $\phi_1(\theta)$ and $\phi_2(\nu)$ of \mathfrak{J}^C by

$$\phi_1(\theta)X = \begin{pmatrix} \theta^4 \xi_1 & \theta x_3 & \theta \bar{x}_2 \\ \theta \bar{x}_3 & \theta^{-2} \xi_2 & \theta^{-2} x_1 \\ \theta x_2 & \theta^{-2} \bar{x}_1 & \theta^{-2} \xi_3 \end{pmatrix}, \quad \phi_2(\nu)X = \begin{pmatrix} \xi_1 & \nu x_3 & \nu^{-1} \bar{x}_2 \\ \nu \bar{x}_3 & \nu^2 \xi_2 & x_1 \\ \nu^{-1} x_2 & \bar{x}_1 & \nu^{-2} \xi_3 \end{pmatrix}, \quad X \in \mathfrak{J}^C,$$

respectively. Then we have that $\phi_1(\theta), \phi_2(\nu) \in (E_6)^\sigma \cap (E_6)^{\sigma'}$, moreover we have that $\phi_2(\nu) \in ((E_6)_{E_1})^{\sigma'}$ and that $\phi_1(\theta), \phi_2(\nu)$ are commutative.

Proof. By straightforward computation, we can also easily prove this lemma. \square

Proposition 4.11.5. We have the following isomorphism: $((E_6)_{E_1})^{\sigma'} \cong (U(1) \times Spin(8))/Z_2$, $Z_2 = \{(1, 1), (-1, \sigma)\}$.

Proof. Let $Spin(8) \cong (F_4)^\sigma \cap (F_4)^{\sigma'} = ((F_4)^\sigma)^{\sigma'} = ((F_4)_{E_1})^{\sigma'} \subset ((E_6)_{E_1})^{\sigma'}$ (Theorem 4.4.5). Now, we define a mapping $\varphi_2 : U(1) \times Spin(8) \rightarrow ((E_6)_{E_1})^{\sigma'}$ by

$$\varphi_2(\nu, \beta) = \phi_2(\nu)\beta.$$

It is clear that $\varphi_2(\nu, \beta) \in ((E_6)_{E_1})^{\sigma'}$ (Lemma 4.11.4). Hence φ_2 is well-defined. Since $\phi_2(\nu)$ commutes with β , φ_2 is a homomorphism. Let $(\nu, \beta) \in \text{Ker } \varphi_2$. Then since we see that $\beta = 1$, we can easily obtain that $\text{Ker } \varphi_2 = \{(1, 1), (-1, \sigma)\}$. Furthermore since $((E_6)_{E_1})^{\sigma'} = (Spin(10))^{\sigma'}$ is connected and $\dim((e_6)_{E_1})^{\sigma'} = 29 = 1 + 28 = \dim(\mathfrak{u}(1) \oplus \mathfrak{so}(8))$ (Lemma 4.11.3), we have that φ_2 is surjection. Therefore we have the isomorphism $((E_6)_{E_1})^{\sigma'} \cong (U(1) \times Spin(8))/Z_2$. \square

Now, we determine the structure of the group $(E_6)^\sigma \cap (E_6)^{\sigma'}$ from Proposition 4.11.5.

Theorem 4.11.6. We have that $(E_6)^\sigma \cap (E_6)^{\sigma'} \cong (U(1) \times U(1) \times Spin(8))/(Z_2 \times Z_4)$, $Z_2 = \{(1, 1, 1), (1, -1, \sigma)\}$, $Z_4 = \{(1, 1, 1), (-i, i, \sigma'), (-1, -1, 1), (i, -i, \sigma')\}$.

Proof. Let $Spin(8) \cong (F_4)^\sigma \cap (F_4)^{\sigma'} \subset (E_6)^\sigma \cap (E_6)^{\sigma'}$ (Theorem 4.4.5). We define a mapping $\varphi_{4116} : U(1) \times U(1) \times Spin(8) \rightarrow (E_6)^\sigma \cap (E_6)^{\sigma'}$ by

$$\varphi_{4116}(\theta, \nu, \beta) = \phi_1(\theta)\phi_2(\nu)\beta.$$

It is clear that $\varphi_{4116}(\theta, \nu, \beta) \in (E_6)^\sigma \cap (E_6)^{\sigma'}$ (Lemma 4.11.4). Hence φ_{4116} is well-defined. Since $\phi_1(\theta), \phi_2(\nu)$ and β commute with each other, it is clear that φ_{4116} is a homomorphism.

We shall show that φ_{4116} is surjection. Let $\alpha \in (E_6)^\sigma \cap (E_6)^{\sigma'}$. Since $\alpha \in (E_6)^\sigma \cap (E_6)^{\sigma'} \subset (E_6)^\sigma$, there exist $\theta \in U(1)$ and $\delta \in Spin(10)$ such that $\alpha = \varphi_{E_3}(\theta, \delta)$ (Theorem 3.3.3). Moreover, from $\alpha = \varphi_{E_3}(\theta, \delta) \in (E_6)^{\sigma'}$, that is, $\sigma' \varphi_{E_3}(\theta, \delta) \sigma' = \varphi_{E_3}(\theta, \delta)$, using $\sigma' \phi_1(\theta) = \phi_1(\theta) \sigma'$ (Lemma 4.11.4), we have $\varphi_{E_3}(\theta, \sigma' \delta \sigma') = \varphi_{E_3}(\theta, \delta)$. Hence it follows that

$$\begin{aligned} \text{(i)} \quad & \begin{cases} \theta = \theta \\ \sigma' \delta \sigma' = \delta, \end{cases} & \text{(ii)} \quad & \begin{cases} \theta = -\theta \\ \sigma' \delta \sigma' = \phi_1(-1)\delta, \end{cases} \\ \text{(iii)} \quad & \begin{cases} \theta = i\theta \\ \sigma' \delta \sigma' = \phi_1(-i)\delta, \end{cases} & \text{(iv)} \quad & \begin{cases} \theta = -i\theta \\ \sigma' \delta \sigma' = \phi_1(i)\delta. \end{cases} \end{aligned}$$

The cases (ii), (iii) and (iv) are impossible because of $\theta = 0$ for $\theta \in U(1)$. In the case (i), from $\sigma' \delta \sigma' = \delta$, we have that $\delta \in (Spin(10))^{\sigma'} \cong ((E_6)_{E_1})^{\sigma'}$. Since there exist $\nu \in U(1)$ and $\beta \in Spin(8)$ such that $\delta = \phi_2(\nu)\beta$ (Proposition 4.11.5), we have that

$$\alpha = \phi_1(\theta)\delta = \phi_1(\theta)\phi_2(\nu)\beta = \varphi(\theta, \nu, \beta).$$

Thus φ is surjection.

From $\text{Ker } \varphi_{E_3} = \{(1, \phi(1)), (-1, \phi(-1)), (i, \phi(-i)), (-i, \phi(i))\}$ and $\text{Ker } \varphi_2 = \{(1, 1), (-1, \sigma)\}$, we can easily obtain that

$$\begin{aligned} \text{Ker } \varphi_{4116} &= \{(1, 1, 1), (1, -1, \sigma), (-1, -1, \sigma), (-1, -1, 1), (i, i, \sigma\sigma'), (i, -i, \sigma'), \\ &\quad (-i, i, \sigma'), (-i, -i, \sigma\sigma')\} \\ &= \{(1, 1, 1), (1, -1, \sigma)\} \times \{(1, 1, 1), (-i, i, \sigma'), (-1, -1, 1), (i, -i, \sigma')\} \\ &\cong \mathbf{Z}_2 \times \mathbf{Z}_4. \end{aligned}$$

Therefore we have the required isomorphism

$$(E_6)^\sigma \cap (E_6)^{\sigma'} \cong (U(1) \times U(1) \times \text{Spin}(8))/(\mathbf{Z}_2 \times \mathbf{Z}_4).$$

□

4.12. Type EIII-IV-IV. In this section, we give a pair of involutive automorphisms λ and $\tilde{\sigma}$.

We define a C -linear transformation δ_9 of \mathfrak{J}^C by

$$\delta_9 X = \begin{pmatrix} \xi_1 & ix_3 & i\bar{x}_2 \\ i\bar{x}_3 & -\xi_2 & -x_1 \\ ix_2 & -\bar{x}_1 & -\xi_3 \end{pmatrix} = D_9 X D_9, D_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}, X \in \mathfrak{J}^C.$$

Then we have that $\delta_9 \in E_6$, $\delta_9^2 = \sigma$, $\delta_9 \sigma = \sigma \delta_9$ and ${}^t \delta_9 = \delta_9$.

Proposition 4.12.1. *The group $(E_6)^\lambda$ is isomorphic to the group $(E_6)^{\lambda\sigma}$: $(E_6)^\lambda \cong (E_6)^{\lambda\sigma}$.*

Proof. We define a mapping $f : (E_6)^\lambda \rightarrow (E_6)^{\lambda\sigma}$ by

$$f(\alpha) = \delta_9 \alpha \delta_9^{-1}.$$

In order to prove this proposition, it is sufficient to show that the mapping f is well-defined. Indeed, it follows from the properties of $\delta_9^2 = \sigma$, $\delta_9 \sigma = \sigma \delta_9$, ${}^t \delta_9 = \delta_9$ and $\lambda(\sigma) = \sigma$ that

$$\begin{aligned} (\lambda\sigma)f(\alpha) &= (\lambda\sigma)(\delta_9 \alpha \delta_9^{-1}) = \lambda(\delta_9 \sigma \alpha \delta_9^{-1}) = \lambda(\delta_9) \lambda(\sigma) \lambda(\alpha) \lambda(\delta_9^{-1}) \\ &= \delta_9^{-1} \sigma \alpha \delta_9 = \delta_9^{-1} (\delta_9)^2 \alpha (\delta_9^{-1} \sigma) = (\delta_9 \alpha \delta_9^{-1}) \sigma \\ &= f(\alpha) \sigma, \end{aligned}$$

that is, $f(\alpha) \in (E_6)^{\lambda\sigma}$. □

From the results of Types EIII, EIV in Table 2 and Proposition 4.12.1, we have the following theorem.

Theorem 4.12.2. *For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \sigma\} \times \{1, \lambda\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(E_6/(E_6)^\sigma, E_6/(E_6)^\lambda, E_6/(E_6)^{\lambda\sigma}) = (E_6/(E_6)^\sigma, E_6/(E_6)^\lambda, E_6/(E_6)^\lambda)$, that is, type (EIII, EIV, EIV), abbreviated as EIII-IV-IV.*

Now, we determine the structure the group $(E_6)^\sigma \cap (E_6)^\lambda$.

Theorem 4.12.3. *We have that $(E_6)^\sigma \cap (E_6)^\lambda \cong \text{Spin}(9)$.*

Proof. We define a mapping $\varphi_{4123} : \text{Spin}(9) \rightarrow (E_6)^\sigma \cap (E_6)^\lambda$ by

$$\varphi_{4123}(\alpha) = \alpha.$$

Since $\text{Spin}(9) \cong (F_4)^\sigma \subset (E_6)^\sigma$ (Theorem 3.2.2) and $\text{Spin}(9) \subset F_4 = (E_6)^\lambda$ (Theorem 3.3.4), it is clear that φ_{4123} is well-defined, a homomorphism and injection.

We shall show that φ_{4123} is surjection. Let $\alpha \in (E_6)^\sigma \cap (E_6)^\lambda$. Since $(E_6)^\sigma \cap (E_6)^\lambda \subset (E_6)^\lambda = F_4$, it is clear that $\alpha \in F_4$. Moreover, from $\alpha \in (E_6)^\sigma$, that is, $\sigma\alpha\sigma = \alpha$, we easily see $\alpha \in (F_4)^\sigma \cong \text{Spin}(9)$. Thus φ_{4123} is surjection.

Therefore we have the required isomorphism

$$(E_6)^\sigma \cap (E_6)^\lambda \cong \text{Spin}(9).$$

□

- $[E_7]$ We study seven types in here.

4.13. **Type EV-V-V.** In this section, we give a pair of involutive inner automorphisms $\tilde{\lambda}\gamma$ and $\iota\tilde{\gamma}_c$.

We define C -linear transformations γ_c of \mathfrak{P}^C by

$$\gamma_c(X, Y, \xi, \eta) = (\gamma_c X, \gamma_c Y, \xi, \eta), (X, Y, \xi, \eta) \in \mathfrak{P}^C,$$

where γ_c of the right hand side are the same ones as $\gamma_c \in G_2 \subset F_4 \subset E_6$. Then we have that $\gamma_c \in E_7$, $\gamma_c^2 = 1$, so $\tilde{\gamma}_c$ of E_7 : $\tilde{\gamma}_c(\alpha) = \gamma_c \alpha \gamma_c, \alpha \in E_7$.

Similarly, for $\delta_3, \delta_4 \in G_2 \subset F_4 \subset E_6$, we have $\delta_3, \delta_4 \in E_7$. Hence, as in E_6 , we easily see that $\delta_3\gamma = \gamma_c\delta_3, \delta_4\gamma = (\gamma\gamma_c)\delta_4$, that is, $\gamma \sim \gamma_c, \gamma \sim \gamma\gamma_c$ in E_7 .

Lemma 4.13.1. *In E_7 , ι is conjugate to λ : $\iota \sim \lambda$.*

Proof. We define a C -linear transformation δ_λ of \mathfrak{P}^C by

$$\delta_\lambda \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \frac{1}{\sqrt{8}} \begin{pmatrix} -(\text{tr}(X)E - 2X) + i(\text{tr}(Y)E - 2Y) - \xi E + i\eta E \\ i(\text{tr}(X)E - 2X) - (\text{tr}(Y)E - 2Y) + i\xi E - \eta E \\ -\text{tr}(X) + i\text{tr}(Y) + \xi - i\eta \\ i\text{tr}(X) - \text{tr}(Y) - i\xi + \eta \end{pmatrix}, \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} \in \mathfrak{P}^C.$$

Then, by straightforward computation, we have $\delta_\lambda \iota = \lambda \delta_\lambda$, that is $\iota \sim \lambda$ in E_7 , moreover $\delta_\lambda \gamma = \gamma \delta_\lambda, \delta_\lambda \gamma_c = \gamma_c \delta_\lambda$. □

Remark. In fact, using $\Phi(0, (i\pi/4)E, (i\pi/4)E, 0) \in \mathfrak{e}_7$, δ_λ is expressed as $\exp \Phi(0, (i\pi/4)E, (i\pi/4)E, 0)$: $\delta_\lambda = \exp \Phi(0, (i\pi/4)E, (i\pi/4)E, 0)$.

Proposition 4.13.2. *The group $(E_7)^{\lambda\gamma}$ is isomorphic to the group $(E_7)^{\iota\gamma_c}$: $(E_7)^{\lambda\gamma} \cong (E_7)^{\iota\gamma_c}$.*

Proof. We define a mapping $f : (E_7)^{\lambda\gamma} \rightarrow (E_7)^{\iota\gamma_c}$ by

$$f(\alpha) = \delta_3 \delta_\lambda^{-1} \alpha \delta_\lambda \delta_3.$$

In order to prove this proposition, it is sufficient to show that the mapping f is well-defined. However, it is almost evident from $\delta_\lambda \iota = \lambda \delta_\lambda, \delta_3 \gamma = \gamma_c \delta_3$. □

For $\theta \in U(1) = \{\theta \in C \mid (\tau^t \theta) \theta = 1\}$, we define a C -linear transformation $\phi(\theta)$ of \mathfrak{P}^C by

$$\phi(\theta)(X, Y, \xi, \eta) = (\theta X, \theta^{-1} Y, \theta^{-3} \xi, \theta^3 \eta), (X, Y, \xi, \eta) \in \mathfrak{P}^C.$$

Then we have $\phi(\theta) \in E_7$, and set $\delta_i = \phi(e^{i\frac{\pi}{4}})$. Needless to say, we see $\delta_i \in E_7$. Besides, the mapping ϕ gives an embedding from $U(1)$ into E_7 .

Lemma 4.13.3. *In E_7 , $\lambda \iota$ is conjugate to $-\lambda$: $\lambda \iota \sim -\lambda$.*

Proof. By using the definition of δ_i above, we can easily obtain $(\lambda \iota) \delta_i = \delta_i(-\lambda)$, that is, $\lambda \iota \sim -\lambda$. □

Proposition 4.13.4. *The group $(E_7)^{\lambda\gamma}$ is isomorphic to the group $(E_7)^{\lambda\gamma\gamma_c}$: $(E_7)^{\lambda\gamma} \cong (E_7)^{\lambda\gamma\gamma_c}$.*

Proof. We define a mapping $g : (E_7)^{\lambda\gamma} \rightarrow (E_7)^{\lambda\gamma\gamma_c}$ by

$$g(\alpha) = \delta_4 \delta_i \alpha \delta_i^{-1} \delta_4,$$

where we remark that δ_4 has the property of $\delta_4(\lambda \iota) = (\lambda \iota) \delta_4$. In order to prove this proposition, it is sufficient to show that the mapping g is well-defined. Indeed, it follows from $(\lambda \iota) \delta_i = \delta_i(-\lambda)$ (Lemma 4.13.3) and the property of δ_4 that

$$\begin{aligned} (\lambda \gamma \gamma_c) g(\alpha) &= (\lambda \gamma \gamma_c) (\delta_4 \delta_i \alpha \delta_i^{-1} \delta_4) = (\lambda \iota) \delta_4 \gamma \delta_i \alpha \delta_i^{-1} \delta_4 = \delta_4(\lambda \iota) \delta_i \gamma \alpha \delta_i^{-1} \delta_4 \\ &= \delta_4 \delta_i(-\lambda) \gamma \alpha \delta_i^{-1} \delta_4 = \delta_4 \delta_i \alpha(-\lambda) \gamma \delta_i^{-1} \delta_4 = \delta_4 \delta_i \alpha(-\lambda) \delta_i^{-1} \gamma \delta_4 \\ &= (\delta_4 \delta_i \alpha \delta_i^{-1}) (\lambda \gamma \delta_4) = (\delta_4 \delta_i \alpha \delta_i^{-1} \delta_4) (\lambda \gamma \gamma_c) = g(\alpha) (\lambda \gamma \gamma_c), \end{aligned}$$

that is, $g(\alpha) \in (E_7)^{\iota\gamma\gamma_c}$. \square

From the result of type EV in Table 2 and Propositions 4.13.2, 4.13.4, we have the following theorem.

Theorem 4.13.5. *For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \lambda\gamma\} \times \{1, \iota\gamma_c\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(E_7/(E_7)^{\lambda\gamma}, E_7/(E_7)^{\iota\gamma_c}, E_7/(E_7)^{(\lambda\gamma)(\iota\gamma_c)}) = (E_7/(E_7)^{\lambda\gamma}, E_7/(E_7)^{\lambda\gamma}, E_7/(E_7)^{\lambda\gamma})$, that is, type (EV, EV, EV), abbreviated as EV-V-V.*

Here, we prove lemma needed in theorem below.

Lemma 4.13.6. *The mapping $\varphi_{E_5} : SU(8) \rightarrow (E_7)^{\lambda\gamma}$ of Theorem 3.4.1 satisfies the following equalities:*

$$\begin{aligned} (1) \quad & \gamma = \varphi_{E_5}(I_2), \quad \gamma_c = \varphi_{E_5}(J), \quad \sigma = \varphi_{E_5}(I_4). \\ (2) \quad & \gamma\varphi_{E_5}(A)\gamma = \varphi_{E_5}(I_2AI_2), \quad \gamma_c\varphi_{E_5}(A)\gamma_c = \varphi_{E_5}(JAJ), \quad \sigma\varphi_{E_5}(A)\sigma = \varphi_{E_5}(I_4AI_4), \\ & \iota\varphi_{E_5}(A)\iota^{-1} = \varphi_{E_5}(\bar{J}\bar{A}J), \end{aligned}$$

where $I_2 = \text{diag}(-1, -1, 1, 1, 1, 1, 1, 1)$, $J = \text{diag}(J, J, J, J)$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $I_4 = \text{diag}(-1, -1, -1, -1, 1, 1, 1, 1)$.

Proof. Since the equalities above are the direct results of [10, Lemma 4.5.4], this proof is omitted. \square

Now, we determine the structure of the group $(E_7)^{\lambda\gamma} \cap (E_7)^{\iota\gamma_c}$.

Theorem 4.13.7. *We have that $(E_7)^{\lambda\gamma} \cap (E_7)^{\iota\gamma_c} \cong SO(8)/\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 = \{E, -E\}$, $\mathbb{Z}_2 = \{1, -1\}$.*

Proof. We define a mapping $SO(8) \times \{1, -1\} \rightarrow (E_7)^{\lambda\gamma} \cap (E_7)^{\iota\gamma_c}$ by

$$\begin{aligned} \varphi_{4137}(B, 1) &= \varphi_{E_5}(B), \\ \varphi_{4137}(B, -1) &= \varphi_{E_5}(B)(-1), \end{aligned}$$

where φ_{E_5} is defined in Theorem 3.4.1. Since the element $-1 \in z(E_7)$ (the center of E_7), it is clear that $\varphi_{4137}(B, 1), \varphi_{4137}(B, -1) \in (E_7)^{\lambda\gamma}$, moreover using $\iota\varphi_{E_5}(A)\iota^{-1} = \varphi_{E_5}(\bar{J}\bar{A}J)$ and $\gamma_c\varphi_{E_5}(A)\gamma_c = \varphi_{E_5}(JAJ)$ (Lemma 4.13.6 (2)), we see that $\varphi_{4137}(B, 1), \varphi_{4137}(B, -1) \in (E_7)^{\iota\gamma_c}$. Hence φ_{4137} is well-defined. Since the mapping φ_{4137} is the restriction of the mapping φ_{E_5} , it is easy to verify that φ_{4137} is a homomorphism.

We shall show that φ_{4137} is surjection. Let $\alpha \in (E_7)^{\lambda\gamma} \cap (E_7)^{\iota\gamma_c}$. Since $\alpha \in (E_7)^{\lambda\gamma} \cap (E_7)^{\iota\gamma_c} \subset (E_7)^{\lambda\gamma}$, there exists $A \in SU(8)$ such that $\alpha = \varphi_{E_5}(A)$ (Theorem 3.4.1). Moreover, from $\alpha = \varphi_{E_5}(A) \in (E_7)^{\iota\gamma_c}$, that is, $(\iota\gamma_c)\varphi_{E_5}(A)(\gamma_c\iota^{-1}) = \varphi_{E_5}(A)$, again using $\iota\varphi_{E_5}(A)\iota^{-1} = \varphi_{E_5}(\bar{J}\bar{A}J)$ and $\gamma_c\varphi_{E_5}(A)\gamma_c = \varphi_{E_5}(JAJ)$, we have $\varphi_{E_5}(\bar{A}) = \varphi_{E_5}(A)$. Hence it follows that

$$\bar{A} = A \quad \text{or} \quad \bar{A} = -A.$$

In the former case, we see $A \in SO(8)$, then set $A = B \in SO(8)$. Hence we have that $\alpha = \varphi_{E_5}(B) = \varphi_{4137}(B)$. In the latter case, we have that $A = e_1B$, $B \in SO(8)$. Hence we have that $\alpha = \varphi_{E_5}(e_1B) = \varphi_{E_5}(e_1E)\varphi_{E_5}(B) = (-1)\varphi_{E_5}(B)$, that is, $\alpha = \varphi_{4137}(B)(-1)$. Thus φ_{4137} is surjection.

Finally, we shall determine $\text{Ker}\varphi_{4137}$. From the definition of kernel, we have that

$$\text{Ker}\varphi_{4137} = \{(B, 1) \mid \varphi_{4137}(B, 1) = 1\} \cup \{(B, -1) \mid \varphi_{4137}(B, -1) = 1\}.$$

In the former case, from $\text{Ker}\varphi_{E_5} = \{E, -E\}$, we can easily obtain that $\{(B, 1) \mid \varphi_{4137}(B, 1) = 1\} = \{(E, 1), (-E, 1)\}$. In the latter case, from $-1 = \varphi_{E_5}(e_1E)$, it is not difficult to see that $\{(B, -1) \mid \varphi_{4137}(B, -1) = 1\} = \{(e_1, -1), (-e_1, -1)\}$. However, since this is contrary to $B \in SO(8)$, this case is impossible. Hence we have $\text{Ker}\varphi_{4137} = \{(E, 1), (-E, 1)\} \cong (\mathbb{Z}_2, 1)$.

Therefore we have the required isomorphism

$$(E_7)^{\lambda\gamma} \cap (E_7)^{\gamma c} \cong SO(8)/\mathbf{Z}_2 \times \mathbf{Z}_2.$$

□

4.14. Type EV-V-VI. In this section, we give a pair of involutive inner automorphisms $\tilde{\lambda}\gamma$ and $\lambda\gamma\sigma$.

Using the inclusion $E_6 \subset E_7$, the C -linear transformation δ_5 used in Section 4.6 is naturally extended to the C -linear transformation of \mathfrak{P}^C . Hence, as in E_6 , since we easily see that $\delta_5\gamma = (\gamma\sigma)\delta_5$ as $\delta_5 \in E_6 \subset E_7$, that is, $\gamma \sim \gamma\sigma$ in E_7 , we have the following proposition.

Proposition 4.14.1. *The group $(E_7)^{\lambda\gamma}$ is isomorphic to the group $(E_7)^{\lambda\gamma\sigma}; (E_7)^{\lambda\gamma} \cong (E_7)^{\lambda\gamma\sigma}$.*

From the result of types EV, EVI in Table 2 and Proposition 4.14.1, we have the following Theorem.

Theorem 4.14.2. *For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \lambda\gamma\} \times \{1, \lambda\gamma\sigma\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(E_7/(E_7)^{\lambda\gamma}, E_7/(E_7)^{\lambda\gamma\sigma}, E_7/(E_7)^{(\lambda\gamma)(\lambda\gamma\sigma)}) = (E_7/(E_7)^{\lambda\gamma}, E_7/(E_7)^{\lambda\gamma}, E_7/(E_7)^{-\sigma}) = (E_7/(E_7)^{\lambda\gamma}, E_7/(E_7)^{\lambda\gamma}, E_7/(E_7)^{\gamma})$, that is, type (EV, EV, EVI), abbreviated as EV-V-VI.*

Here, we prove proposition needed and make some preparations for the theorem below.

Proposition 4.14.3. *We have the following isomorphism: $S(U(4) \times U(4)) \cong (U(1) \times SU(4) \times SU(4))/\mathbf{Z}_4$, $\mathbf{Z}_4 = \{(1, E, E), (-1, -E, -E), (e_1, -e_1E, e_1E), (-e_1, e_1E, -e_1E)\}$.*

Proof. We define a mapping $g : U(1) \times SU(4) \times SU(4) \rightarrow S(U(4) \times U(4))$ by

$$g_4(a, A, B) = \begin{pmatrix} aA & 0 \\ 0 & a^{-1}B \end{pmatrix}.$$

Then we easily see that g_4 is well-defined and a epimorphism. By straightforward computation, $\text{Ker } g_4$ is obtained as follows:

$$\begin{aligned} \text{Ker } g_4 &= \{(a, A, B) \in U(1) \times SU(4) \times SU(4) \mid g_4(a, A, B) = E\} \\ &= \{(a, A, B) \in U(1) \times SU(4) \times SU(4) \mid aA = a^{-1}B = E\} \\ &= \{(a, a^{-1}E, aE) \in U(1) \times SU(4) \times SU(4) \mid a = \pm 1, \pm e_1\} \\ &= \{(1, E, E), (-1, -E, -E), (e_1, -e_1E, e_1E), (-e_1, e_1E, -e_1E)\} \cong \mathbf{Z}_4. \end{aligned}$$

Therefore we have the required isomorphism

$$S(U(4) \times U(4)) \cong (U(1) \times SU(4) \times SU(4))/\mathbf{Z}_4.$$

We define some element $\varepsilon \in (E_7)^{\lambda\gamma}$ by

$$\varepsilon = \varphi_{\text{E5}}(J'),$$

□

where $J' = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \in SU(8)$, $E = \text{diag}(1, 1, 1, 1)$. Then we easily see $\varepsilon^2 = 1$.

Consider a group $\mathbf{Z}_2 = \{1, \varepsilon\}$. Then the group $\mathbf{Z}_2 = \{1, \varepsilon\}$ acts on the group $S(U(4) \times U(4))$ by

$$\varepsilon(A) = J'AJ', (J')^2 = E (= \text{diag}(1, 1, 1, 1, 1, 1, 1, 1)),$$

and let $S(U(4) \times U(4)) \rtimes \mathbf{Z}_2$ be the semi-direct product of $S(U(4) \times U(4))$ and \mathbf{Z}_2 with this action.

Now, we determine the structure of the group $(E_7)^{\lambda\gamma} \cap (E_7)^{\lambda\gamma\sigma}$.

Theorem 4.14.4. *We have that $(E_7)^{\lambda\gamma} \cap (E_7)^{\lambda\gamma\sigma} \cong (U(1) \times SU(4) \times SU(4))/(\mathbf{Z}_2 \times \mathbf{Z}_4) \rtimes \mathbf{Z}_2$, $\mathbf{Z}_2 = \{(1, E, E), (1, -E, -E)\}$, $\mathbf{Z}_4 = \{(1, E, E), (-1, -E, -E), (e_1, -e_1E, e_1E), (-e_1, e_1E, -e_1E)\}$, $\mathbf{Z}_2 = \{1, \varepsilon\}$.*

Proof. We define a mapping $\varphi_{4144} : (U(1) \times SU(4) \times SU(4)) \rtimes \{1, \varepsilon\} \rightarrow (E_7)^{\lambda\gamma} \cap (E_7)^{\lambda\gamma\sigma}$ by

$$\begin{aligned}\varphi_{4144}((a, A, B), 1) &= \varphi_{\text{Es}}(g_4(a, A, B)), \\ \varphi_{4144}((a, A, B), \varepsilon) &= \varphi_{\text{Es}}(g_4(a, A, B))\varepsilon,\end{aligned}$$

where g_4 is defined in Proposition 4.14.3 above. Since the mapping φ_{4144} is the restriction of the mapping φ_{Es} and $\varepsilon \in (E_7)^{\lambda\gamma}$, it is clear that $\varphi_{4144}((a, A, B), 1), \varphi_{4144}((a, A, B), \varepsilon) \in (E_7)^{\lambda\gamma}$, moreover using $\sigma\varphi_{\text{Es}}(L)\sigma = \varphi_{\text{Es}}(I_4LI_4)$, $L \in SU(8)$ (Lemma 4.13.6 (2)), it is easily to verify that $\varphi_{4144}((a, A, B), 1), \varphi_{4144}((a, A, B), \varepsilon) \in (E_7)^{\lambda\gamma\sigma}$. Hence φ_{4144} is well-defined. Using $\varepsilon = \varphi_{\text{Es}}(J')$, we can confirm that φ_{4144} is a homomorphism. Indeed, we show the case of $\varphi_{4144}((a_1, A_1, B_1), \varepsilon)\varphi_{4144}((a_2, A_2, B_2), 1) = \varphi_{4144}(((a_1, A_1, B_1), \varepsilon)((a_2, A_2, B_2), 1))$ as example. For the left hand side of this equality, we have that

$$\begin{aligned}\varphi_{4144}((a_1, A_1, B_1), \varepsilon)\varphi_{4144}((a_2, A_2, B_2), 1) &= \varphi_{\text{Es}}(g_4(a_1, A_1, B_1))\varepsilon\varphi_{\text{Es}}(g_4(a_2, A_2, B_2)) \\ &= \varphi_{\text{Es}}(g_4(a_1, A_1, B_1))\varphi_{\text{Es}}(J')\varphi_{\text{Es}}(g_4(a_2, A_2, B_2)) \\ &= \varphi_{\text{Es}}(g_4(a_1, A_1, B_1)J'(g_4(a_2, A_2, B_2))) \\ &= \varphi_{\text{Es}}(g_4(a_1, A_1, B_1)J'(g_4(a_2, A_2, B_2)J'J')) \\ &= \varphi_{\text{Es}}(g_4(a_1, A_1, B_1)J'(g_4(a_2, A_2, B_2)J')\varphi_{\text{Es}}(J')) \\ &= \varphi_{\text{Es}}(g_4(a_1, A_1, B_1)J'(g_4(a_2, A_2, B_2)J')\varepsilon) \\ &= \varphi_{\text{Es}}(g_4(a_1, A_1, B_1)(g_4(a_2^{-1}, B_2, A_2))\varepsilon) \\ &= \varphi_{\text{Es}}(g_4(a_1a_2^{-1}, A_1B_2, B_1A_2))\varepsilon \\ &= \varphi_{4144}((a_1a_2^{-1}, A_1B_2, B_1A_2), \varepsilon).\end{aligned}$$

On the other hand, since the action of ε to the group $S(U(4) \times U(4))$ is $\varepsilon\left(\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}\right) = \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix}$, ε acts on the group $U(1) \times SU(4) \times SU(4)$ as follows:

$$\varepsilon(a, A, B) = (a^{-1}, B, A).$$

Hence, for the right hand side of same one, we have that

$$\begin{aligned}\varphi_{4144}(((a_1, A_1, B_1), \varepsilon)((a_2, A_2, B_2), 1)) &= \varphi_{4144}((a_1, A_1, B_1)\varepsilon(a_2, A_2, B_2), \varepsilon) \\ &= \varphi_{4144}((a_1, A_1, B_1)(a_2^{-1}, B_2, A_2), \varepsilon) \\ &= \varphi_{4144}((a_1a_2^{-1}, A_1B_2, B_1A_2), \varepsilon).\end{aligned}$$

Similarly, the other cases are shown.

We shall show that φ_{4144} is surjection. Let $\alpha \in (E_7)^{\lambda\gamma} \cap (E_7)^{\lambda\gamma\sigma}$. Since $(E_7)^{\lambda\gamma} \cap (E_7)^{\lambda\gamma\sigma} \subset (E_7)^{\lambda\gamma}$, there exists $L \in SU(8)$ such that $\alpha = \varphi_{\text{Es}}(L)$ (Theorem 3.4.1). Moreover, from $\alpha = \varphi_{\text{Es}}(L) \in (E_7)^{\lambda\gamma\sigma}$, that is, $(\lambda\gamma\sigma)\varphi_{\text{Es}}(L)(\sigma\gamma\lambda^{-1}) = \varphi_{\text{Es}}(L)$, using $\sigma\varphi_{\text{Es}}(L)\sigma = \varphi_{\text{Es}}(I_4LI_4)$ (Lemma 4.13.6 (2)), we have $\varphi_{\text{Es}}(I_4LI_4) = \varphi_{\text{Es}}(L)$. Hence it follows that

$$I_4LI_4 = L \quad \text{or} \quad I_4LI_4 = -L.$$

In the former case, we see that $L \in S(U(4) \times U(4))$. Hence, there exist $a \in U(1), A, B \in SU(4)$ such that $L = g_4(a, A, B)$ (Proposition 4.14.3). Thus we have $\alpha = \varphi_{\text{Es}}(g_4(a, A, B)) = \varphi_{4144}((a, A, B), 1)$. In the latter case, we take L as form $L = MJ'$, $M \in S(U(4) \times U(4))$, $J' = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$, $E = \text{diag}(1, 1, 1, 1)$. Hence, in a similar way as above, we have that $\alpha = \varphi_{\text{Es}}(g_4(a, A, B)J') = \varphi_{\text{Es}}(g_4(a, A, B))\varphi_{\text{Es}}(J') = \varphi_{\text{Es}}(g_4(a, A, B))\varepsilon = \varphi_{4144}((a, A, B), \varepsilon)$. Hence φ_{4144} is surjection.

Finally, we shall determine $\text{Ker } \varphi_{4144}$. From $\text{Ker } \varphi_{\text{E5}} = \{E, -E\}$, we can easily obtain that

$$\begin{aligned} \text{Ker } \varphi_{4144} &= \{((a, A, B), 1) \mid \varphi_{4144}((a, A, B), 1) = 1\} \cup \{((a, A, B), \varepsilon) \mid \varphi_{4144}((a, A, B), \varepsilon) = 1\} \\ &= \{((a, A, B), 1) \mid \varphi_{\text{E5}}(g_4(a, A, B)) = 1\} \cup \{((a, A, B), \varepsilon) \mid \varphi_{\text{E5}}(g_4(a, A, B))\varepsilon = 1\} \\ &= \{((a, A, B), 1) \mid aA = a^{-1}B = \pm E\} \cup \{((a, A, B), \varepsilon) \mid aA = a^{-1}B = 0\} \\ &= \{(a, a^{-1}E, aE), 1\}, (a, -a^{-1}E, -aE), 1 \mid a^4 = 1\} \cup \phi \\ &= \{((1, E, E), 1), ((1, -E, -E), 1)\} \\ &\quad \times \{((1, E, E), 1), ((-1, -E, -E), 1), ((e_1, -e_1E, e_1E), 1), ((-e_1, e_1E, -e_1E), 1)\} \\ &\cong (\mathbb{Z}_2 \times \mathbb{Z}_4, 1). \end{aligned}$$

Therefore we have the required isomorphism

$$(E_7)^{\lambda\gamma} \cap (E_7)^{\lambda\gamma\sigma} \cong (U(1) \times SU(4) \times SU(4)) / (\mathbb{Z}_2 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2.$$

□

4.15. Type EV-V-VII. In this section, we give a pair of involutive inner automorphisms $\tilde{\lambda}\gamma$ and $\iota\tilde{\lambda}\gamma$.

We have the following proposition which is the direct result of Lemmas 4.13.1, 4.13.3.

Proposition 4.15.1. *The group $(E_7)^{\lambda\gamma}$ is isomorphic to the group $(E_7)^{\iota\lambda\gamma}$: $(E_7)^{\lambda\gamma} \cong (E_7)^{\iota\lambda\gamma}$.*

From the result of types EV, EVII in Table 2 and Propositions 4.15.1, we have the following theorem.

Theorem 4.15.2. *For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \lambda\gamma\} \times \{1, \iota\lambda\gamma\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(E_7/(E_7)^{\lambda\gamma}, E_7/(E_7)^{\iota\lambda\gamma}, E_7/(E_7)^{(\lambda\gamma)(\iota\lambda\gamma)}) = (E_7/(E_7)^{\lambda\gamma}, E_7/(E_7)^{\lambda\gamma}, E_7/(E_7)^{-\iota}) = (E_7/(E_7)^{\lambda\gamma}, E_7/(E_7)^{\lambda\gamma}, E_7/(E_7)^{\iota})$, that is, type (EV, EV, EVII), abbreviated as EV-V-VII.*

Here, we prove lemma needed in theorem below.

Lemma 4.15.3. *The C -linear transformation $\phi(\theta)$ defined in Section 4.13 satisfies the following equalities:*

$$\lambda\phi(\theta)\lambda = \phi(\theta^{-1}), \quad \gamma\phi(\theta)\gamma = \phi(\theta).$$

Proof. By using the definition of λ, γ and $\phi(\theta)$ (Sections 3.4, 4.13), it is easily to verify those. □

Now, we determine the structure of the group $(E_7)^{\lambda\gamma} \cap (E_7)^{\iota\lambda\gamma}$.

Theorem 4.15.4. *We have that $(E_7)^{\lambda\gamma} \cap (E_7)^{\iota\lambda\gamma} \cong (Sp(4)/\mathbb{Z}_2) \times \mathbb{Z}_2$, $\mathbb{Z}_2 = \{E, -E\}$, $\mathbb{Z}_2 = \{1, -1\}$.*

Proof. We define a mapping $\varphi_{4154} : Sp(4) \times \{1, -1\} \rightarrow (E_7)^{\lambda\gamma} \cap (E_7)^{\iota\lambda\gamma}$ by

$$\begin{aligned} \varphi_{4154}(A, 1) &= \varphi_{\text{E1}}(A), \\ \varphi_{4154}(A, -1) &= \varphi_{\text{E1}}(A)(-1), \end{aligned}$$

where φ_{E1} is defined in Theorem 3.3.1. (Remark. The element $\varphi_{\text{E1}}(A) \in (E_6)^{\lambda\gamma}$ is identified as elements of the group E_7 .) Since the mapping φ_{4154} is the restriction of the mapping φ_{E1} , it is clear that φ_{4154} is well-defined and a homomorphism.

We shall show that φ_{4154} is surjection. Let $\alpha \in (E_7)^{\lambda\gamma} \cap (E_7)^{\iota\lambda\gamma}$. Since $\lambda\gamma\alpha\gamma\lambda^{-1} = \alpha$ and $\iota(\lambda\gamma\alpha\gamma\lambda^{-1})\iota^{-1} = \alpha$, we have that $\alpha \in (E_7)^{\iota}$. Hence, there exist $\theta \in U(1)$ and $\beta \in E_6$ such that $\alpha = \varphi_{\text{E7}}(\theta, \beta)$ (Theorem 3.4.3). Moreover, from $\alpha \in (E_7)^{\lambda\gamma}$, that is, $(\lambda\gamma)\varphi_{\text{E7}}(\theta, \beta)(\gamma\lambda^{-1}) = \varphi_{\text{E7}}(\theta, \beta)$, using $\lambda\phi(\theta)\lambda^{-1} = \phi(\theta^{-1})$ and $\gamma\phi(\theta)\gamma = \phi(\theta)$ (Lemma

4.15.3), we have $\varphi_{E_7}(\theta^{-1}, \lambda\gamma\beta\gamma\lambda^{-1}) = \varphi_{E_7}(\theta, \beta)$. Then, as the argument above, we also see $\alpha \in (E_7)^{\lambda\gamma}$. Hence, it follows that

$$(i) \begin{cases} \theta^{-1} = \theta \\ \lambda\gamma\beta\gamma\lambda^{-1} = \beta, \end{cases} \quad (ii) \begin{cases} \theta^{-1} = \omega\theta \\ \lambda\gamma\beta\gamma\lambda^{-1} = \phi(\omega^2)\beta, \end{cases} \quad (iii) \begin{cases} \theta^{-1} = \omega^2\theta \\ \lambda\gamma\beta\gamma\lambda^{-1} = \phi(\omega)\beta, \end{cases}$$

where $\omega \in C$, $\omega^3 = 1$, $\omega \neq 1$. For these cases above, we have the following results.

Case (i). We have that $\theta = -1$ or $\theta = 1$ and $\beta \in (E_6)^{\lambda\gamma}$. Hence, in the case of $\theta = 1$, there exists $A \in Sp(4)$ such that $\alpha = \varphi_{E_7}(1, \beta) = \beta = \varphi_{E_1}(A) = \varphi_{4154}(A, 1)$ (Theorem 3.3.1), and in the case of $\theta = -1$, similarly there exists $A \in Sp(4)$ such that $\alpha = \varphi_{E_7}(-1, \beta) = \phi(-1)\beta = (-1)\beta = \varphi_{E_1}(A)(-1) = \varphi_{4154}(A, -1)$.

Case (ii). We have that $\theta = -\omega$ or $\theta = \omega$ and $\beta = \phi(\omega^2)\beta', \beta' \in (E_6)^{\lambda\gamma}$. Hence, in the case of $\theta = \omega$, there exists $A' \in Sp(4)$ such that $\alpha = \varphi_{E_7}(\omega, \phi(\omega^2)\beta') = \phi(\omega)(\phi(\omega^2)\beta') = \beta' = \varphi_{E_1}(A') = \varphi_{4154}(A', 1)$ (Theorem 3.3.1), and in the case of $\theta = -\omega$, similarly there exists $A' \in Sp(4)$ such that $\alpha = \varphi_{E_7}(-\omega, \phi(\omega^2)\beta') = \phi(-\omega)(\phi(\omega^2)\beta') = (-1)\beta' = \varphi_{E_1}(A')(-1) = \varphi_{4154}(A', -1)$. As a result, this case is reduced to Case (i).

Case (iii). We have that $\theta = -\omega^2$ or $\theta = \omega^2$ and $\beta \in \phi(\omega)\beta', \beta' \in (E_6)^{\lambda\gamma}$. Hence we have the same result as Case (ii), that is, this case is also reduced to Case (i).

Thus φ_{4154} is surjection. Finally, we shall determine $\text{Ker } \varphi_{4154}$. From $\text{Ker } \varphi_{E_1} = \{E, -E\}$, we can easily obtain that

$$\begin{aligned} \text{Ker } \varphi_{4154} &= \{(A, 1) \mid \varphi_{4154}(A, 1) = 1\} \cup \{(A, 1) \mid \varphi_{4154}(A, -1) = 1\} \\ &= \{(A, 1) \mid \varphi_{E_1}(A) = 1\} \cup \{(A, -1) \mid \varphi_{E_1}(A)(-1) = 1\} \\ &= \{(E, 1), (-E, 1)\} \cup \phi \\ &= \{(E, 1), (-E, 1)\} \cong (\mathbb{Z}_2, 1). \end{aligned}$$

Therefore we have the required isomorphism

$$(E_7)^{\lambda\gamma} \cap (E_7)^{\lambda\gamma} \cong (Sp(4)/\mathbb{Z}_2) \times \mathbb{Z}_2.$$

□

4.16. Type EV-VI-VII. In this section, we give a pair of involutive inner automorphisms $\tilde{\lambda}\gamma$ and $\tilde{\gamma}$.

We have the following proposition which is the direct result of Lemma 4.13.1.

Proposition 4.16.1. *The group $(E_7)^\lambda$ is isomorphic to the group $(E_7)^\gamma : (E_7)^\lambda \cong (E_7)^\gamma$.*

From the result of types EV, EVI, EVII in Table 2 and Proposition 4.16.1, we have the following theorem.

Theorem 4.16.2. *For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \lambda\gamma\} \times \{1, \gamma\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(E_7/(E_7)^{\lambda\gamma}, E_7/(E_7)^\gamma, E_7/(E_7)^{(\lambda\gamma)(\gamma)}) = (E_7/(E_7)^{\lambda\gamma}, E_7/(E_7)^\gamma, E_7/(E_7)^\lambda) = (E_7/(E_7)^{\lambda\gamma}, E_7/(E_7)^\gamma, E_7/(E_7)^\gamma)$, that is, type (EV, EVI, EVII), abbreviated as EV-VI-VII.*

Here, we prove proposition needed in theorem below.

Proposition 4.16.3. *We have the following isomorphism: $S(U(2) \times U(6)) \cong (U(1) \times SU(2) \times SU(6))/\mathbb{Z}_{12}$, $\mathbb{Z}_{12} = \{(e^{e_1 \frac{2\pi}{12}k}, e^{-e_1 \frac{12\pi}{12}k}E, e^{e_1 \frac{4\pi}{12}k}E) \mid k = 0, 1, 2, \dots, 11\}$.*

Proof. We define a mapping $f_6 : U(1) \times SU(2) \times SU(6) \rightarrow S(U(2) \times U(6))$ by

$$f_6(a, A, B) = \begin{pmatrix} a^6 A & 0 \\ 0 & a^{-2} B \end{pmatrix}.$$

Then we easily see that f_6 is well-defined and a epimorphism. By straightforward computation, $\text{Ker } f_6$ is obtained as follows:

$$\begin{aligned} \text{Ker } f_6 &= \{(a, A, B) \in U(1) \times SU(2) \times SU(6) \mid f_6(a, A, B) = E\} \\ &= \{(a, A, B) \in U(1) \times SU(2) \times SU(6) \mid a^6 A = a^{-2} B = E\} \\ &= \{(a, a^{-6} E, a^2 E) \in U(1) \times SU(2) \times SU(6) \mid a^{12} = 1\} \\ &= \{(e^{e^{i1} \frac{2\pi}{12} k}, e^{-e^{i1} \frac{12\pi}{12} k} E, e^{e^{i1} \frac{4\pi}{12} k} E) \mid k = 0, 1, 2, \dots, 11\} \cong \mathbf{Z}_{12}. \end{aligned}$$

Therefore we have the required isomorphism

$$S(U(2) \times U(6)) \cong (U(1) \times SU(2) \times SU(6))/\mathbf{Z}_{12}.$$

□

Now, we determine the structure of the group $(E_7)^{\lambda\gamma} \cap (E_7)^\gamma$.

Theorem 4.16.4. *We have that $(E_7)^{\lambda\gamma} \cap (E_7)^\gamma \cong (U(1) \times SU(2) \times SU(6))/\mathbf{Z}_{24}$, $\mathbf{Z}_{24} = \{(a, a^6 E, a^{-2} E) \mid a = e^{e^{i1} \frac{2\pi}{12} k}, k = 0, 1, 2, \dots, 23\}$.*

Proof. We define a mapping $\varphi_{4164} : U(1) \times SU(2) \times SU(6) \rightarrow (E_7)^{\lambda\gamma} \cap (E_7)^\gamma$ by

$$\varphi_{4164}(f_6(a, A, B)) = \varphi_{\text{Es}}(f_6(a, A, B)).$$

Since the mapping φ_{4164} is the restriction of the mapping φ_{Es} , it is clear that $\varphi_{4164}(f_6(a, A, B)) \in (E_7)^{\lambda\gamma}$, and using $\gamma\varphi_{\text{Es}}(L)\gamma = \varphi_{\text{Es}}(I_2 L I_2)$, $L \in SU(8)$ (Lemma 4.13.6 (2)), it is easily to verify that $\varphi_{4164}(f_6(a, A, B)) \in (E_7)^\gamma$. Hence, φ_{4164} is well-defined. Again, since the mapping φ_{4164} is the restriction of the mapping φ_{Es} , it is clear that φ_{4164} is a homomorphism.

We shall show that φ_{4164} is surjection. Let $\alpha \in (E_7)^{\lambda\gamma} \cap (E_7)^\gamma$. From $(E_7)^{\lambda\gamma} \cap (E_7)^\gamma \subset (E_7)^{\lambda\gamma}$, there exists $L \in SU(8)$ such that $\alpha = \varphi_{\text{Es}}(L)$ (Theorem 3.4.1). Moreover, from $\alpha \in (E_7)^\gamma$, that is, $\gamma\varphi_{\text{Es}}(L)\gamma = \varphi_{\text{Es}}(L)$, again using $\gamma\varphi_{\text{Es}}(L)\gamma = \varphi_{\text{Es}}(I_2 L I_2)$ (Lemma 4.13.6 (2)), we have that $\varphi_{\text{Es}}(I_2 L I_2) = \varphi_{\text{Es}}(L)$. Hence, it follows that

$$I_2 L I_2 = L \quad \text{or} \quad I_2 L I_2 = -L.$$

In the former case, we see that $L \in S(U(2) \times U(6))$. Hence, there exist $a \in U(1)$, $A \in SU(2)$ and $SU(6)$ such that $L = f_6(a, A, B)$ (Proposition 4.16.3). Thus we have that $\alpha = \varphi_{\text{Es}}(L) = \varphi_{\text{Es}}(f_6(a, A, B)) = \varphi_{4164}(a, A, B)$. In the latter case, as the former case, we can also find the explicit form of $L \in SU(8)$ as follows:

$$L = \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}, \quad C \in M(2, 6, \mathbf{C}), D \in M(6, 2, \mathbf{C}).$$

This case is impossible because of $\det L = 0$. Thus φ_{4164} is surjection.

Finally, we shall determine $\text{Ker } \varphi_{4164}$. From $\text{Ker } \varphi_{\text{Es}} = \{E, -E\}$, we can easily obtain that

$$\begin{aligned} \text{Ker } \varphi_{4164} &= \{(a, A, B) \in U(1) \times SU(2) \times SU(6) \mid \varphi_{4164}(a, A, B) = 1\} \\ &= \{(a, A, B) \in U(1) \times SU(2) \times SU(6) \mid \varphi_{\text{Es}}(f_6(a, A, B)) = 1\} \\ &= \{(a, A, B) \in U(1) \times SU(2) \times SU(6) \mid f_6(a, A, B) = E, f_6(a, A, B) = -E\} \\ &= \{(a, a^6 E, a^{-2} E) \in U(1) \times SU(2) \times SU(6) \mid a^{12} = 1, a^{12} = -1\} \\ &= \{(a, a^6 E, a^{-2} E) \mid a = e^{e^{i1} \frac{2\pi}{12} k}, k = 0, 1, 2, \dots, 23\} \cong \mathbf{Z}_{24}. \end{aligned}$$

Therefore we have the required isomorphism

$$(E_7)^{\lambda\gamma} \cap (E_7)^\gamma \cong (U(1) \times SU(2) \times SU(6))/\mathbf{Z}_{24}.$$

□

4.17. Type EVI-VI-VI. In this section, there exist two cases with this type.

4.17.1. We begin from the first case: $\mathfrak{k} = \mathfrak{so}(8) \oplus \mathfrak{so}(4) \oplus i\mathbf{R}$.

In the first case, we give a pair of involutive inner automorphisms $\tilde{\gamma}$ and $-\tilde{\sigma}$. We remark that $-\tilde{\sigma}$ is same as $\tilde{\sigma}$ because of $-1 \in z(E_7)$ (the center of E_7). Again, we state $\gamma \sim \gamma\sigma, \gamma \sim -\sigma$ in E_7 as mentioned in Sections 4.14, 3.4, respectively.

From the result of type EVI in Table 2 and $\gamma \sim \gamma\sigma, \gamma \sim -\sigma$, we have the following theorem.

Remark. From $-1 \in z(E_7)$, it is clear that $(E_7)^{-\gamma\sigma} = (E_7)^{\gamma\sigma}$.

Theorem 4.17.1-1 For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \gamma\} \times \{1, -\sigma\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(E_7/(E_7)^\gamma, E_7/(E_7)^{-\sigma}, E_7/(E_7)^{\gamma(-\sigma)}) = (E_7/(E_7)^\gamma, E_7/(E_7)^\gamma, E_7/(E_7)^{\gamma\sigma}) = (E_7/(E_7)^\gamma, E_7/(E_7)^\gamma, E_7/(E_7)^\gamma)$, that is, type (EVI, EVI, EVI), abbreviated as EVI-VI-VI.

Now, we determine the structure of the group $(E_7)^\gamma \cap (E_7)^{-\sigma} = (E_7)^\gamma \cap (E_7)^\sigma$.

Theorem 4.17.1-2 We have that $(E_7)^\gamma \cap (E_7)^{-\sigma} = (E_7)^\gamma \cap (E_7)^\sigma \cong (SU(2) \times Spin(4) \times Spin(8))/(Z_2 \times Z_2)$, $Z_2 \times Z_2 = \{(E, 1, 1), (E, \sigma, \sigma)\} \times \{(E, 1, 1), (-E, \gamma, -\sigma\gamma)\}$.

Proof. Let $Spin(4) \cong (((E_7)^{\kappa, \mu})^\gamma)_{\tilde{F}_1(h), \tilde{E}_1, \tilde{E}_{-1}, \tilde{E}_{23}}$ and $Spin(8) \cong (((E_7)^{\kappa, \mu})^\gamma)_{\tilde{F}_1(he_4)}$. Since both of the groups $Spin(4)$ and $Spin(8)$ are the subgroups of $Spin(12) \cong (E_7)^{\kappa, \mu}$, we can define a mapping $\varphi_{4171-2} : SU(2) \times Spin(4) \times Spin(8) \rightarrow (E_7)^\gamma \cap (E_7)^\sigma$ as the restriction of the mapping φ_{E_6} as follows:

$$\varphi_{4171-2}(A, \beta_4, \beta_8) = \phi_2(A)\beta_4\beta_8.$$

Then this mapping induces the required isomorphism (see [6, Theorem 3.23] in detail). \square

4.17.2. Next, we study the second case: $\mathfrak{k} = \mathfrak{u}(6) \oplus i\mathbf{R}$.

In the second case, we give a pair of involutive inner automorphisms $\tilde{\gamma}$ and $\tilde{\gamma}_H$.

We define C -linear transformations γ_H of \mathfrak{P}^C by

$$\gamma_H(X, Y, \xi, \eta) = (\gamma_H X, \gamma_H Y, \xi, \eta),$$

where γ_H of the right hand side are the same ones as $\gamma_H \in G_2 \subset F_4 \subset E_6$. Then we have that $\gamma_H \in E_7, \gamma_H^2 = 1$, so γ_H induce involutive inner automorphism $\tilde{\gamma}_H$ of E_7 : $\tilde{\gamma}_H(\alpha) = \gamma_H \alpha \gamma_H \in E_7$.

Similarly, for $\delta_1, \delta_2 \in G_2 \subset F_4 \subset E_6$, we have $\delta_1, \delta_2 \in E_7$. Hence, as in E_6 , since we easily see that $\delta_1\gamma = \gamma_H\delta_1, \delta_2\gamma = (\gamma\gamma_H)\delta_2$, that is, $\gamma \sim \gamma_H, \gamma \sim \gamma\gamma_H$ in E_7 , we have the following proposition.

Proposition 4.17.2-1 The group $(E_7)^\gamma$ is isomorphic to both of the groups $(E_7)^{\gamma_H}$ and $(E_7)^{\gamma\gamma_H}$: $(E_7)^\gamma \cong (E_7)^{\gamma_H} \cong (E_7)^{\gamma\gamma_H}$.

From the result of type EVI in Table 2 and Proposition 4.17.2-1, we have the following theorem.

Theorem 4.17.2-2 For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \gamma\} \times \{1, \gamma_H\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(E_7/(E_7)^\gamma, E_7/(E_7)^{\gamma_H}, E_7/(E_7)^{\gamma\gamma_H}) = (E_7/(E_7)^\gamma, E_7/(E_7)^\gamma, E_7/(E_7)^\gamma)$, that is, type (EVI, EVI, EVI), abbreviated as EVI-VI-VI.

Here, we prove proposition needed and make some preparations for the theorem below.

First, using identifying $(\mathfrak{Z}_C)^C \oplus M(3, C)^C$ with \mathfrak{Z}^C , we identify $(\mathfrak{P}_C)^C \oplus (M(3, C)^C \oplus M(3, C)^C)$ with \mathfrak{P}^C by

$$(X, Y, \xi, \eta) + (M, N) = (X + M, Y + N, \xi, \eta),$$

where $(\mathfrak{P}_C)^C = (\mathfrak{Z}_C)^C \oplus (\mathfrak{Z}_C)^C \oplus C \oplus C$. (As for identifying $(\mathfrak{Z}_C)^C \oplus M(3, C)^C$ with \mathfrak{Z}^C and the definition of $(\mathfrak{Z}_C)^C$, see [7, Sections 2.2, 2.3].)

We often denote any element of $M(3, \mathbf{C})^C$ by $(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$, where $\mathbf{m}_k \in (\mathbf{C}^3)^C, k = 1, 2, 3$, moreover denote any element of $(\mathfrak{P}_C)^C$ by (X, Y, ξ, η) as above. (Remark. we often denote any element of \mathfrak{P}^C by same one.)

We define a C -linear transformation w of $(\mathfrak{P}_C)^C \oplus (M(3, \mathbf{C})^C \oplus M(3, \mathbf{C})^C) = \mathfrak{P}^C$ by

$$w((X, Y, \xi, \eta) + (M, N)) = (X, Y, \xi, \eta) + (\omega_1 M, \omega_1 N),$$

$$(X, Y, \xi, \eta) + (M, N) \in (\mathfrak{P}_C)^C \oplus (M(3, \mathbf{C})^C \oplus M(3, \mathbf{C})^C) = \mathfrak{P}^C,$$

where $\omega_1 M = (\omega_1 \mathbf{m}_1, \omega_1 \mathbf{m}_2, \omega_1 \mathbf{m}_3)$, $\omega_1 \in \mathbf{C}, \omega_1^3 = 1, \omega_1 \neq 1$, so is $\omega_1 N$.

Besides, w is defined as the C -linear transformation of $\mathbf{C} \oplus \mathbf{C}^3 = \mathfrak{C}$ as follows:

$$w(a + \mathbf{m}) = a + \omega_1 \mathbf{m}, \quad a + \mathbf{m} \in \mathbf{C} \oplus \mathbf{C}^3 = \mathfrak{C}.$$

Then we have that $w \in G_2, w^3 = 1$. Hence, using the inclusion $G_2 \subset F_4 \subset E_6 \subset E_7$, w induces inner automorphism \tilde{w} of order 3 in E_7 : $\tilde{w}(\alpha) = w^{-1} \alpha w, \alpha \in E_7$.

Proposition 4.17.2-3 *We have the following isomorphism: $(E_7)^w \cong (SU(3) \times SU(6)) / \mathbf{Z}_3, \mathbf{Z}_3 = \{(E, E), (\omega_1 E, \omega_1 E), (\omega_1^2 E, \omega_1^2 E)\}$.*

Proof. We define a mapping $\varphi_w : SU(3) \times SU(6) \rightarrow (E_7)^w$ by

$$\varphi_w(D, A)P = f^{-1}((D, A)(fP)), \quad P \in \mathfrak{P}^C.$$

Then this mapping induces the required isomorphism (see [10, Section 4.13] in detail). \square

By identifying $(\mathfrak{P}_C)^C \oplus M(3, \mathbf{C})^C$ with \mathfrak{Z}^C , the C -linear transformation γ, γ_H and γ_C of \mathfrak{Z}^C naturally act on $(\mathfrak{P}_C)^C \oplus M(3, \mathbf{C})^C = \mathfrak{Z}^C$. Hence, using the inclusion $E_6 \subset E_7$, the C -linear transformations γ, γ_H and γ_C of $(\mathfrak{P}_C)^C \oplus M(3, \mathbf{C})^C = \mathfrak{Z}^C$ are naturally extended to the C -linear transformations of $(\mathfrak{P}_C)^C \oplus M(3, \mathbf{C})^C \oplus M(3, \mathbf{C})^C = \mathfrak{P}^C$ as follows:

$$\gamma((X, Y, \xi, \eta) + (M, N)) = (X, Y, \xi, \eta) + (\gamma M, \gamma N),$$

$$\gamma_H((X, Y, \xi, \eta) + (M, N)) = (X, Y, \xi, \eta) + (\gamma_H M, \gamma_H N),$$

$$\gamma_C((X, Y, \xi, \eta) + (M, N)) = (\overline{X}, \overline{Y}, \xi, \eta) + (\overline{M}, \overline{N}),$$

where $\gamma M = \gamma(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3) = (\gamma \mathbf{m}_1, \gamma \mathbf{m}_2, \gamma \mathbf{m}_3)$, and so is $\gamma_H M$. In addition, for $\mathbf{m} = (m_1, m_2, m_3) \in \mathbf{C}^3$, $\gamma \mathbf{m}$ and $\gamma_H \mathbf{m}$ are defined by $(m_1, -m_2, -m_3)$ and $(-m_1, m_2, -m_3)$: $\gamma \mathbf{m} = (m_1, -m_2, -m_3), \gamma_H \mathbf{m} = (-m_1, m_2, -m_3)$, respectively.

Consider a group $\mathcal{Z}_2 = \{1, \gamma_C\}$. Then the group $\mathcal{Z}_2 = \{1, \gamma_C\}$ acts on the group $U(1) \times U(1) \times SU(6)$ by

$$\gamma_C(p, q, A) = (\overline{p}, \overline{q}, (\text{Ad} J_3) \overline{A})$$

and let $(U(1) \times U(1) \times SU(6)) \rtimes \mathcal{Z}_2$ be the semi-direct product of $U(1) \times U(1) \times SU(6)$ and \mathcal{Z}_2 with this action.

Now, we determine the structure of the group $(E_7)^\gamma \cap (E_7)^{\gamma_H}$.

Theorem 4.17.2-4 *We have that $(E_7)^\gamma \cap (E_7)^{\gamma_H} \cong (U(1) \times U(1) \times SU(6)) / \mathbf{Z}_3 \rtimes \mathcal{Z}_2, \mathbf{Z}_3 = \{(1, 1, E), (\omega_1, \omega_1, \omega_1 E), (\omega_1^2, \omega_1^2, \omega_1^2 E)\}, \mathcal{Z}_2 = \{1, \gamma_C\}$, where $\omega_1 \in \mathbf{C}, \omega_1^3 = 1, \omega_1 \neq 1$.*

Proof. We define a mapping $\varphi_{4172-4} : (U(1) \times U(1) \times SU(6)) \rtimes \{1, \gamma_C\} \rightarrow (E_7)^\gamma \cap (E_7)^{\gamma_H}$ by

$$\varphi_{4172-4}((p, q, A), 1) = \varphi_w(D(p, q), A),$$

$$\varphi_{4172-4}((p, q, A), \gamma_C) = \varphi_w(D(p, q), A) \gamma_C,$$

where $D(p, q) = \text{diag}(p, q, \overline{pq}) \in SU(3)$ and φ_w is defined in Proposition 4.17.2-2.

Then this mapping induces the required isomorphism (see [7, Theorem 2.4.3] in detail). \square

4.18. Type EVI-VII-VII. In this section, we give a pair of involutive inner automorphisms $-\sigma$ and $\tilde{\iota}$.

Lemma 4.18.1. *In E_7 , ι is conjugate to $-\sigma\iota$: $\iota \sim -\sigma\iota$.*

Proof. We define a C -linear transformation δ_{10} of \mathfrak{P}^C by

$$\delta_{10} \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} (1-p_1)X - 2E_1 \times Y + \eta E_1 \\ 2E_1 \times X + (1-p_1)Y + \xi E_1 \\ (E_1, Y) \\ (-E_1, X) \end{pmatrix}, \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} \in \mathfrak{P}^C,$$

where p_1 is defined by $p_1(X) = (X, E_1) + 4E_1 \times (E_1 \times X)$, $X \in \mathfrak{J}^C$. Then, by straightforward computation, we have that $\delta_{10} \in E_7$, $\delta_{10}\iota = (-\iota\sigma)\delta_{10}$, that is, $\iota \sim -\sigma\iota$ in E_7 . \square

We have the following proposition which is the direct result of Lemma 4.18.1.

Proposition 4.18.2. *The group $(E_7)^\iota$ is isomorphic to the group $(E_7)^{-\sigma\iota}$: $(E_7)^\iota \cong (E_7)^{-\sigma\iota}$.*

From the result of types EVI, EVII in Table 2 and Proposition 4.18.2, we have the following theorem.

Theorem 4.18.3. *For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, -\sigma\} \times \{1, \iota\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(E_7/(E_7)^{-\sigma}, E_7/(E_7)^\iota, E_7/(E_7)^{(-\sigma)\iota}) = (E_7/(E_7)^\sigma, E_7/(E_7)^\iota, E_7/(E_7)^\iota)$, that is, type (EVI, EVII, EVII), abbreviated as EVI-VII-VII.*

Now, we determine the structure of the group $(E_7)^{-\sigma} \cap (E_7)^\iota$.

Theorem 4.18.4. *We have that $(E_7)^{-\sigma} \cap (E_7)^\iota \cong (U(1) \times U(1) \times Spin(10))/(Z_4 \times Z_3)$, $Z_4 = \{(1, 1, 1), (1, -1, -\sigma), (1, -i, \sigma\phi_1(i)), (1, -i, \phi_1(i))\}$, $Z_3 = \{(1, 1, 1), (\omega, \omega, 1), (\omega^2, \omega^2, 1)\}$, where $\omega \in C$, $\omega^3 = 1$, $\omega \neq 1$.*

Proof. Let $U(1) = \{a \in C \mid (\tau a)a = 1\}$ and $Spin(10) \cong (E_6)_{E_1} = \{\alpha \in E_6 \mid \alpha E_1 = E_1\}$. We define a mapping $\varphi_{4184} : U(1) \times U(1) \times Spin(10) \rightarrow (E_7)^{-\sigma} \cap (E_7)^\iota = (E_7)^\sigma \cap (E_7)^\iota$ by

$$\varphi_{4184}(\theta, a, \delta) = \phi(\theta)\phi_1(a)\delta,$$

where ϕ, ϕ_1 are defined in Theorems 3.4.3, 3.3.3, respectively. It is clear that $\varphi_{4184}(\theta, a, \delta) \in (E_7)^\iota$, moreover since $\sigma\phi(\theta)\sigma = \phi(\theta)$ and $\phi_1(a)\delta \in (E_6)^\sigma$, it is easily to verify that $\varphi_{4184}(\theta, a, \delta) \in (E_7)^\sigma = (E_7)^{-\sigma}$. Hence φ_{4184} is well-defined. Since $\phi(\theta)$ commutes with $\phi_1(a)$ and δ each other and moreover $\phi_1(a)$ commutes with δ , we easily see that φ_{4184} a homomorphism.

We shall show that φ_{4184} is surjection. Let $\alpha \in (E_7)^{-\sigma} \cap (E_7)^\iota$. Since $(E_7)^{-\sigma} \cap (E_7)^\iota \subset (E_7)^\iota$, there exist $\theta \in U(1)$ and $\beta \in E_6$ such that $\alpha = \varphi_{E_7}(\theta, \beta)$ (Theorem 3.4.3). Moreover, from $\alpha \in (E_7)^{-\sigma} = (E_7)^\sigma$, that is, $\sigma\varphi_{E_7}(\theta, \beta)\sigma = \varphi_{E_7}(\theta, \beta)$, we have $\varphi_{E_7}(\theta, \sigma\beta\sigma) = \varphi_{E_7}(\theta, \beta)$. Hence, it follows that

$$(i) \begin{cases} \theta = \theta \\ \sigma\beta\sigma = \beta, \end{cases} \quad (ii) \begin{cases} \theta = \omega\theta \\ \sigma\beta\sigma = \phi(\omega^2)\beta, \end{cases} \quad (iii) \begin{cases} \theta = \omega^2\theta \\ \sigma\beta\sigma = \phi(\omega)\beta. \end{cases}$$

Then we can easily confirm that (ii) and (iii) are impossible because of $\theta = 0$. In the case (i), we have that $\beta \in (E_6)^\sigma \cong (U(1) \times Spin(10))/Z_4$. Hence, from Theorem 3.3.3, there exist $a \in U(1)$ and $\delta \in Spin(10)$ such that $\beta = \varphi_{E_3}(a, \delta) = \phi_1(a)\delta$. Thus φ_{4184} is surjection.

Finally, we shall determine $\text{Ker } \varphi_{4184}$. From $\text{Ker } \varphi_{E_3} = \{(1, \phi(1)), (-1, \phi(-1)), (i, \phi(-i)), (-i, \phi(i))\}$, we can easily obtain that

$$\begin{aligned} \text{Ker } \varphi_{4184} &= \{(\theta, a, \delta) \in U(1) \times U(1) \times \text{Spin}(10) \mid \varphi_{4184}(\theta, a, \delta) = 1\} \\ &= \{(\theta, a, \delta) \in U(1) \times U(1) \times \text{Spin}(10) \mid \phi(\theta)\phi_1(a)\delta = 1\} \\ &= \{(\theta, a, \delta) \in U(1) \times U(1) \times \text{Spin}(10) \mid \theta^3 = 1, a^4 = 1, \delta = \phi_1(a^{-1})\} \\ &= \{(1, 1, 1), (1, -1, -\sigma), (1, -i, \sigma\phi_1(i)), (1, -i, \phi_1(i)), \\ &\quad (\omega, \omega^{\frac{1}{4}}, \phi_1(i)), (\omega, \omega^{-\frac{1}{4}}, \sigma\phi_1(i)), (\omega, i\omega^{\frac{1}{4}}, 1), (\omega, -i\omega^{\frac{1}{4}}, \sigma), \\ &\quad (\omega^2, \omega^{\frac{1}{2}}, \sigma), (\omega^2, -\omega^{-\frac{1}{2}}, 1), (\omega^2, i\omega^{\frac{1}{2}}, \phi_1(i)), (\omega^2, -i\omega^{\frac{1}{2}}, \sigma\phi_1(i))\} \\ &= \{(1, 1, 1), (1, -1, -\sigma), (1, -i, \sigma\phi_1(i)), (1, -i, \phi_1(i))\} \\ &\quad \times \{(1, 1, 1), (\omega, \omega, 1), (\omega^2, \omega^2, 1)\} \cong \mathbf{Z}_4 \times \mathbf{Z}_3. \end{aligned}$$

Therefore we have the required isomorphism

$$(E_7)^{-\sigma} \cap (E_7)^t \cong (U(1) \times U(1) \times \text{Spin}(10))/(\mathbf{Z}_4 \times \mathbf{Z}_3).$$

□

4.19. Type EVII-VII-VII. In this section, we give a pair of involutive inner automorphisms \tilde{t} and $\tilde{\lambda}$.

Proposition 4.19.1. *The group $(E_7)^t$ is isomorphic to both of the groups $(E_7)^\lambda$ and $(E_7)^{\iota\lambda}$: $(E_7)^t \cong (E_7)^\lambda \cong (E_7)^{\iota\lambda}$.*

Proof. First, we have $(E_7)^t \cong (E_7)^\lambda$ as the direct result of Lemma 4.13.1.

Next, we define a mapping $g : (E_7)^{\iota\lambda} \rightarrow (E_7)^t$ by

$$g(\alpha) = (\delta_\lambda^{-1} \delta_i^{-1}) \alpha (\delta_i \delta_\lambda),$$

where both of δ_λ and δ_i are defined in Section 4.13. In order to prove this proposition, it is sufficient to show that the mapping g is well-defined. Indeed, it follows from $(\lambda\iota)\delta_i = \delta_i(\lambda)$ and $\lambda\delta_\lambda = \delta_\lambda\iota$ that

$$\begin{aligned} \iota g(\alpha) &= \iota((\delta_\lambda^{-1} \delta_i^{-1}) \alpha (\delta_i \delta_\lambda)) = \delta_\lambda^{-1} ((\lambda\delta_i^{-1}) \alpha (\delta_i \delta_\lambda)) = (\delta_\lambda^{-1} \delta_i^{-1}) ((-\lambda\iota) \alpha (\delta_i \delta_\lambda)) \\ &= (\delta_\lambda^{-1} \delta_i^{-1}) (\alpha (-\lambda\iota) (\delta_i \delta_\lambda)) = (\delta_\lambda^{-1} \delta_i^{-1}) (\alpha \delta_i \lambda \delta_\lambda) = ((\delta_\lambda^{-1} \delta_i^{-1}) \alpha (\delta_i \delta_\lambda)) \iota \\ &= g(\alpha) \iota, \end{aligned}$$

that is, $g(\alpha) \in (E_7)^t$. □

From the result of type EVII in Table 2 and Proposition 4.19.1, we have the following theorem

Theorem 4.19.2. *For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \iota\} \times \{1, \lambda\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(E_7/(E_7)^t, E_7/(E_7)^\lambda, E_7/(E_7)^{\iota\lambda}) = (E_7/(E_7)^t, E_7/(E_7)^t, E_7/(E_7)^t)$, that is, type (EVII, EVII, EVII), abbreviated as EVII-VII-VII.*

Now, we determine the structure of the group $(E_7)^t \cap (E_7)^\lambda$.

Theorem 4.19.3. *We have that $(E_7)^t \cap (E_7)^\lambda \cong F_4 \times \mathbf{Z}_2$, $\mathbf{Z}_2 = \{1, -1\}$.*

Proof. We define a mapping $\varphi_{4193} : F_4 \times \{1, -1\}$ by

$$\begin{aligned} \varphi_{4193}(\alpha, 1) &= \varphi_{E_7}(1, \alpha), \\ \varphi_{4193}(\alpha, -1) &= \varphi_{E_7}(-1, \alpha), \end{aligned}$$

where φ_{E_7} is defined in Theorem 3.4.3. Since the mapping φ_{4193} is the restriction of the mapping φ_{E_7} , it is clear that φ_{4193} is well-defined and a homomorphism.

We shall show that φ_{4193} is surjection. Let $\alpha \in (E_7)^t \cap (E_7)^\lambda$. From $(E_7)^t \cap (E_7)^\lambda \subset (E_7)^t$, there exist $\theta \in U(1)$ and $\beta \in E_6$ such that $\alpha = \varphi_{E_7}(\theta, \beta) = \phi(\theta)\beta$ (Theorem 3.4.3). Moreover, from $\alpha \in (E_7)^\lambda$, that is, $\lambda\varphi_{E_7}(\theta, \beta)\lambda^{-1} = \varphi_{E_7}(\theta, \beta)$, using $\lambda\phi(\theta)\lambda^{-1} = \phi(\theta^{-1})$ (Lemma 4.15.3), we have that $\varphi_{E_7}(\theta^{-1}, \lambda\beta\lambda^{-1}) = \varphi_{E_7}(\theta, \beta)$. Hence, it follows that

$$(i) \begin{cases} \theta^{-1} = \theta \\ \lambda\beta\lambda^{-1} = \beta, \end{cases} \quad (ii) \begin{cases} \theta^{-1} = \omega\theta \\ \lambda\beta\lambda^{-1} = \phi(\omega^2)\beta, \end{cases} \quad (iii) \begin{cases} \theta^{-1} = \omega^2\theta \\ \lambda\beta\lambda^{-1} = \phi(\omega)\beta. \end{cases}$$

Case (i). We see that $\theta = \pm 1$ and $\beta \in F_4 \cong (E_6)^\lambda$. Hence, in the case of $\theta = 1$, there exist $1 \in U(1)$ and $\beta \in F_4$ such that $\alpha = \phi(1)\beta = \beta$, that is, $\alpha = \varphi_{E_7}(1, \alpha) = \varphi_{4193}(\alpha, 1)$. Similarly, in the case of $\theta = -1$, we have that $\alpha = \varphi_{E_7}(-1, \alpha) = \varphi_{4193}(\alpha, -1)$.

Case (ii). We see that $\theta = \pm\omega$ and $\beta = \phi(\omega^2)\beta', \beta' \in F_4$. Hence, in the case of $\theta = \omega$, there exist $\omega \in U(1)$ and $\beta = \phi(\omega^2)\beta'$ such that $\alpha = \phi(\omega)\phi(\omega^2)\beta' = \beta'$, that is, $\alpha = \varphi_{E_7}(1, \alpha) = \varphi_{4193}(\alpha, 1)$. Similarly, in the case of $\theta = -\omega$, we have that $\alpha = \varphi_{E_7}(-1, \alpha) = \varphi_{4193}(\alpha, -1)$. Thus this case is reduced to Case (i).

Case (iii). We see that $\theta = \pm\omega^2$ and $\beta = \phi(\omega)\beta', \beta' \in F_4$. As in Case (ii), this case is also reduced to Case (i).

Finally, we shall determine the $\text{Ker } \varphi_{4193}$, however it is easily obtained that $\text{Ker } \varphi_{4193} = (\{1\}, 1)$.

Therefore we have the required isomorphism

$$(E_7)^t \cap (E_7)^\lambda \cong F_4 \times \mathbb{Z}_2.$$

□

- $[E_8]$ We study four types in here.

4.20. Type EVIII-VIII-VIII. In this section, we give a pair of involutive inner automorphisms $\tilde{\sigma}$ and $\tilde{\sigma}'$, where C -linear transformations σ, σ' of \mathfrak{e}_8^C are defined below.

We define C -linear transformations σ, σ' of \mathfrak{e}_8^C by

$$\sigma(\Phi, P, Q, r, s, t) = (\sigma\Phi\sigma, \sigma P, \sigma Q, r, s, t),$$

$$\sigma'(\Phi, P, Q, r, s, t) = (\sigma'\Phi\sigma', \sigma'P, \sigma'Q, r, s, t), \quad (\Phi, P, Q, r, s, t) \in \mathfrak{e}_8^C,$$

where σ, σ' of the right hand side are same ones as $\sigma, \sigma' \in F_4 \subset E_6 \subset E_7$. Then we have that that $\sigma, \sigma' \in E_8, \sigma^2 = \sigma'^2 = 1$. Hence σ, σ' induce involutive inner automorphisms $\tilde{\sigma}, \tilde{\sigma}'$ of E_8 : $\tilde{\sigma}(\alpha) = \sigma\alpha\sigma, \tilde{\sigma}'(\alpha) = \sigma'\alpha\sigma', \alpha \in E_8$.

Lemma 4.20.1. (1) *The Lie algebra $(\mathfrak{e}_8)^\sigma$ of the group $(E_8)^\sigma$ is given by*

$$(\mathfrak{e}_8)^\sigma = \{(\Phi, \tau\lambda Q, Q, r, s, -\tau s) \in \mathfrak{e}_8 \mid \Phi \in (\mathfrak{e}_7)^\sigma \cong \mathfrak{su}(2) \oplus \mathfrak{so}(12), Q \in (\mathfrak{P}^C)_\sigma, r \in i\mathbf{R}, s \in C\}.$$

(2) *The Lie algebra $(\mathfrak{e}_8)^{\lambda\omega\gamma}$ of the group $(E_8)^{\lambda\omega\gamma}$ is given by*

$$(\mathfrak{e}_8)^{\lambda\omega\gamma} = \{(\Phi, \lambda\gamma Q, Q, 0, s, -s) \in \mathfrak{e}_8 \mid \Phi \in (\mathfrak{e}_7)^{\lambda\gamma} = (\mathfrak{e}_7)^{\tau\gamma} \cong \mathfrak{su}(8), Q \in (\mathfrak{P}^C)_{\tau\gamma}, s \in \mathbf{R}\}.$$

In particular, we have that

$$\begin{aligned} \dim((\mathfrak{e}_8)^\sigma) &= (3 + 66) + ((3 + 8) \times 2 + 2) \times 2 + 1 + 1 \times 2 = 120 \\ &= 63 + (3 + (4 \times 3) \times 2) + 2 + 1 = \dim((\mathfrak{e}_8)^{\lambda\omega\gamma}). \end{aligned}$$

Proof. By straightforward computation, we can easily prove this lemma. □

From Lemma 4.20.1 and [13, Lemma 5.3.3], we have the following proposition.

Proposition 4.20.2. *The group $(E_8)^\sigma$ is isomorphic to the group $(E_8)^{\lambda\omega\gamma}$: $(E_8)^\sigma \cong (E_8)^{\lambda\omega\gamma}$ ($\cong Ss(16)$).*

Remark. The author can not find any element $\delta \in E_8$ which gives the conjugation: $\delta\sigma = (\lambda_\omega\gamma)\delta$.

Here, using the inclusion $F_4 \subset E_6 \subset E_7 \subset E_8$, the C -linear transformations δ_6, δ_7 defined in the proof of Lemma 4.4.1 are naturally extended to the C -linear transformations of \mathfrak{e}_8^C . Hence, as in E_6 , since we easily see that $\delta_6\sigma = \sigma'\delta_6, \delta_7\sigma = (\sigma\sigma')\delta_7$ as $\delta_6, \delta_7 \in F_4 \subset E_6 \subset E_7 \subset E_8$, that is, $\sigma \sim \sigma', \sigma \sim \sigma\sigma'$ in E_8 , we have the following proposition.

Proposition 4.20.3. *The group $(E_8)^\sigma$ is isomorphic to both of the groups $(E_8)^{\sigma'}$ and $(E_8)^{\sigma\sigma'}$: $(E_8)^\sigma \cong (E_8)^{\sigma'} \cong (E_8)^{\sigma\sigma'}$.*

From the result of type EVIII in Table 2 and Propositions 4.20.2, 4.20.3, we have the following theorem.

Theorem 4.20.4. *For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \sigma\} \times \{1, \sigma'\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(E_8/(E_8)^\sigma, E_8/(E_8)^{\sigma'}, E_8/(E_8)^{\sigma\sigma'}) = (E_8/(E_8)^\sigma, E_8/(E_8)^\sigma, E_8/(E_8)^\sigma)$, that is, type (EVIII, EVIII, EVIII), abbreviated as EVIII-VIII-VIII.*

Now, we determine the structure of the group $(E_8)^\sigma \cap (E_8)^{\sigma'}$.

Theorem 4.20.5. *We have that $(E_8)^\sigma \cap (E_8)^{\sigma'} \cong (Spin(8) \times Spin(8))/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, $\mathbb{Z}_2 = \{(1, 1), (\sigma, \sigma)\}$, $\mathbb{Z}_2 = \{(1, 1), (\sigma', \sigma')\}$.*

Proof. This proof is omitted (see [8, Theorem 7.1]). The purpose of [8] is to prove the this theorem). \square

4.21. Type EVIII-VIII-IX. In this section, we use a pair of involutive inner automorphisms $\lambda_\omega\gamma$ and $\lambda_\omega\gamma\nu$.

Lemma 4.21.1. *In E_8 , $\lambda_\omega\gamma\nu$ is conjugate to $\lambda_\omega\gamma$: $\lambda_\omega\gamma\nu \sim \lambda_\omega\gamma$.*

Proof. We define a C -linear transformation δ_ν of \mathfrak{e}_8^C by

$$\delta_\nu(\Phi, P, Q, r, s, t) = (\Phi, iP, -iQ, r, -s, -t), (\Phi, P, Q, r, s, t) \in \mathfrak{e}_8^C.$$

Then we have that $\delta_\nu \in E_8, \delta_\nu^2 = \nu$ and $\delta_\nu(\lambda_\omega\gamma\nu) = (\lambda_\omega\gamma)\delta_\nu$: $\lambda_\omega\gamma\nu \sim \lambda_\omega\gamma$ in E_8 , moreover that $\delta_\nu\lambda = \lambda\delta_\nu$. \square

We have the following proposition which is the direct result of Lemma 4.21.1.

Proposition 4.21.2. *The group $(E_8)^{\lambda_\omega\gamma\nu}$ is isomorphic to the group $(E_8)^{\lambda_\omega\gamma}$: $(E_8)^{\lambda_\omega\gamma\nu} \cong (E_8)^{\lambda_\omega\gamma}$.*

From the results of types EXIII, EIX in Table 2 and Proposition 4.21.1, we have the following theorem.

Theorem 4.21.3. *For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \lambda_\omega\gamma\} \times \{1, \lambda_\omega\gamma\nu\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(E_8/(E_8)^{\lambda_\omega\gamma}, E_8/(E_8)^{\lambda_\omega\gamma\nu}, E_8/(E_8)^{(\lambda_\omega\gamma)(\lambda_\omega\gamma\nu)}) = (E_8/(E_8)^{\lambda_\omega\gamma}, E_8/(E_8)^{\lambda_\omega\gamma}, E_8/(E_8)^\nu)$, that is, type (EVIII, EVIII, EIX), abbreviated as EVIII-VIII-IX.*

Here, we prove lemma needed in theorem below.

Lemma 4.21.4. *The mapping $\phi_3 : SU(2) \rightarrow E_8$ of Theorem 3.5.2 satisfies*

- (1) $\delta_\nu = \phi_3(iI)$.
- (2) $\lambda_\omega\phi_3(A)\lambda_\omega^{-1} = \iota_\omega\phi_3(A)\iota_\omega^{-1} = \phi_3({}^tA^{-1})$.
- (3) $\sigma\phi_3(A)\sigma = \phi_3(A), \gamma\phi_3(A)\gamma = \phi_3(A)$,

where $iI = \text{diag}(i, -i) \in SU(2)$.

Proof. By straightforward computation, we can easily prove this lemma. (The C -linear transformation ι_ω of $(e_8)^C$ is defined in Section 4.23. As for the definition of the mapping ϕ_3 , see [10, Theorem 5.7.4].) \square

Consider a group $\mathcal{Z}_2 = \{1, \rho_v\}$, where $\rho_v = \delta_v \iota$. Then the group \mathcal{Z}_2 acts on the group $SO(2) \times SU(8)$ by

$$\rho_v(A, B) = ((iI)A(iI)^{-1}, J\bar{B}J^{-1}),$$

where $J = \text{diag}(J_1, J_1, J_1, J_1) \in M(8, \mathbf{R})$, $J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and let $(SO(2) \times SU(8)) \rtimes \mathcal{Z}_2$ be the semi-direct product $SO(2) \times SU(8)$ and \mathcal{Z}_2 with this action.

Now, we determine the structure of the group $(E_8)^{\lambda_\omega \gamma} \cap (E_8)^{\lambda_\omega \gamma v}$.

Theorem 4.21.5. *We have that $(E_8)^{\lambda_\omega \gamma} \cap (E_8)^{\lambda_\omega \gamma v} \cong (SO(2) \times SU(8))/\mathbf{Z}_4 \rtimes \mathcal{Z}_2$, $\mathbf{Z}_4 = \{(E, E), (E, -E), (-E, e_1 E), (-E, -e_1 E)\}$, $\mathcal{Z}_2 = \{1, \rho_v\}$.*

Proof. We define a mapping $\varphi_{4215} : (SO(2) \times SU(8)) \rtimes \{1, \rho_v\} \rightarrow (E_8)^{\lambda_\omega \gamma} \cap (E_8)^{\lambda_\omega \gamma v}$ by

$$\begin{aligned} \varphi_{4215}((A, B), 1) &= \varphi_{E_9}(A, \varphi_{E_8}(B)), \\ \varphi_{4215}((A, B), \rho) &= \varphi_{E_9}(A, \varphi_{E_8}(B))\rho_v, \end{aligned}$$

where $\varphi_{E_9}, \varphi_{E_8}$ are defined in Theorems 3.5.2, 3.4.1, respectively. From $\lambda_\omega \gamma \phi_3(A) \gamma \lambda_\omega^{-1} = \phi(A)$, $A \in SO(2)$ (Lemma 4.21.4 (3)) and $\varphi_{E_8}(B) \in (E_7)^{\iota \gamma}$ (Theorem 3.4.1), it is easily to verify that φ_{4215} is well-defined. By straightforward computation, we can confirm that φ_{4215} is a homomorphism. Indeed, we show that the case of $\varphi_{4215}((A, B), \rho_v) \varphi_{4215}((C, D), 1) = \varphi_{4215}((A, B), \rho_v)((C, D), 1)$ as example. For the left hand side of this equality, we have that

$$\begin{aligned} \varphi_{4215}((A, B), \rho_v) \varphi_{4215}((C, D), 1) &= \varphi_{E_9}(A, \varphi_{E_8}(B))\rho_v \varphi_{E_9}(C, \varphi_{E_8}(D)) \\ &= \phi_3(A)\varphi_{E_8}(B)\rho_v \phi_3(C)\varphi_{E_8}(D). \end{aligned}$$

On the other hand, for the right hand side of same one, using $\delta_v = \phi_3(iI), \iota \varphi_{E_8}(A) \iota^{-1} = \varphi_{E_8}(J\bar{A}J)$ (Lemmas 4.21.4 (1), 4.13.6 (2)), we have that

$$\begin{aligned} \varphi_{4215}((A, B), \rho_v)((C, D), 1) &= \varphi_{4215}(((A, B)\rho(C, D))), \rho_v) \\ &= \varphi_{4215}(((A, B)((iI)C(iI)^{-1}, J\bar{D}J^{-1})), \rho_v) \\ &= \varphi_{4215}((A(iI)C(iI)^{-1}, B J\bar{D}J^{-1}), \rho_v) \\ &= \phi_3(A(iI)C(iI)^{-1})\varphi_{E_8}(B J\bar{D}J^{-1})\rho_v \\ &= \phi_3(A(iI)C(iI)^{-1})\varphi_{E_8}(B J\bar{D}J^{-1})(\delta_v \iota)(\leftarrow J^{-1} = -J) \\ &= \phi_3(A)\delta_v \phi_3(C)\delta_v^{-1} \varphi_{E_8}(B)\iota \varphi_{E_8}(D)\iota^{-1}(\delta_v \iota) \\ &= \phi_3(A)\delta_v \phi_3(C)\delta_v^{-1} \varphi_{E_8}(B)\iota \varphi_{E_8}(D)\delta_v \\ &= \phi_3(A)\delta_v \phi_3(C)\varphi_{E_8}(B)\iota \varphi_{E_8}(D) \\ &= \phi_3(A)\delta_v \varphi_{E_8}(B)\phi_3(C)\iota \varphi_{E_8}(D)(\leftarrow \varphi_{E_8}(B), \iota \in E_7) \\ &= \phi_3(A)\varphi_{E_8}(B)(\delta_v \iota) \phi_3(C)\varphi_{E_8}(D) \\ &= \phi_3(A)\varphi_{E_8}(B)\rho_v \phi_3(C)\varphi_{E_8}(D). \end{aligned}$$

Similarly, the other cases are shown.

We shall show that φ_{4215} is surjection. Let $\alpha \in (E_8)^{\lambda_\omega \gamma} \cap (E_8)^{\lambda_\omega \gamma v}$. From $(E_8)^{\lambda_\omega \gamma} \cap (E_8)^{\lambda_\omega \gamma v} \subset (E_8)^{(\lambda_\omega \gamma)(\lambda_\omega \gamma v)} = (E_8)^v$, there exist $A \in SU(2)$ and $\beta \in E_7$ such that $\alpha = \varphi_{E_9}(A, \beta)$ (Theorem 3.5.2). Moreover, from $\alpha = \varphi_{E_9}(A, \beta) \in (E_8)^{\lambda_\omega \gamma}$, that is, $\lambda_\omega \gamma \varphi_{E_9}(A, \beta) \gamma \lambda_\omega^{-1} = \varphi_{E_9}(A, \beta)$, using $\lambda_\omega \gamma \phi_3(A) \gamma \lambda_\omega^{-1} = \phi_3({}^t A^{-1})$ (Lemma 4.21.4 (2)), we have that $\varphi_{E_9}({}^t A^{-1}, \lambda \gamma \beta \gamma \lambda^{-1}) = \varphi_{E_9}(A, \beta)$. (Remark. For $\alpha \in (E_8)^v$, that $\alpha \in (E_8)^{\lambda_\omega \gamma}$ implies that $\alpha \in (E_8)^{\lambda_\omega \gamma v}$.)

Hence, it follows that

$$\begin{cases} {}^tA^{-1} = A \\ \lambda\gamma\beta\gamma\lambda^{-1} = \beta \end{cases} \quad \text{or} \quad \begin{cases} {}^tA^{-1} = -A \\ \lambda\gamma\beta\gamma\lambda^{-1} = -\beta. \end{cases}$$

In the former case, we see that $A \in SO(2)$ and $\beta \in (E_7)^{\lambda\gamma} \cong SU(8)/\mathbf{Z}_2$. Hence, there exists $B \in SU(8)$ such that $\beta = \varphi_{\text{Es}}(B)$ (Theorem 3.4.1). Thus we have that $\alpha = \varphi_{\text{E9}}(A, \beta) = \varphi_{\text{E9}}(A, \varphi_{\text{Es}}(B)) = \varphi_{4215}((A, B), 1)$. In the latter case, we see that $A = A'(iI), A' \in SO(2)$ and $\beta = \beta'\iota, \beta' \in (E_7)^{\lambda\gamma}$. Hence, in a similar way as the former case, we have that

$$\begin{aligned} \alpha &= \varphi_{\text{E9}}(A, \beta) = \varphi_{\text{E9}}(A'(iI), \beta'\iota) = \phi_3(A'(iI))(\beta'\iota) = \phi_3(A')\phi_3(iI)(\beta'\iota) \\ &= \phi_3(A')\phi_3(iI)\beta'\iota = \phi_3(A')\beta'(\phi_3(iI)\iota) = \varphi_{\text{E9}}(A', \beta')(\delta_v\iota) = \varphi_{\text{E9}}(A', \varphi_{\text{Es}}(B'))\rho_v \\ &= \varphi_{4215}((A', B'), \rho_v). \end{aligned}$$

Thus φ_{4214} is surjection.

Finally, we shall determine $\text{Ker } \varphi_{4215}$. From the definition of kernel, it is as follows:

$$\begin{aligned} \text{Ker } \varphi_{4215} &= \{((A, B), 1) \mid \varphi_{4215}((A, B), 1) = 1\} \cup \{((A, B), \rho_v) \mid \varphi_{4215}((A, B), \rho_v) = 1\} \\ &= \{((A, B), 1) \mid \varphi_{\text{E9}}(A, \varphi_{\text{Es}}(B)) = 1\} \cup \{((A, B), \rho) \mid \varphi_{\text{E9}}(A, \varphi_{\text{Es}}(B))\rho_v = 1\}. \end{aligned}$$

Here, for the left hand side case, we have that

$$\begin{aligned} &\{((A, B), 1) \mid \varphi_{\text{E9}}(A, \varphi_{\text{Es}}(B)) = 1\} \\ &= \{((A, B), 1) \mid A = \pm E, \varphi_{\text{Es}}(B) = \pm 1\} \\ &= \{((E, E), 1), ((E, -E), 1), ((-E, -e_1E), 1), ((-E, e_1E), 1)\}. \end{aligned}$$

On the other hand, for the right hand side case, since $\varphi_{\text{E9}}(A, \varphi_{\text{Es}}(B))\rho_v = 1$, we suppose that

$$\varphi_{\text{E9}}(A, \varphi_{\text{Es}}(B))\rho_v(0, 0, 0, 0, 1, 0) = (0, 0, 0, 0, 1, 0), \text{ where } (0, 0, 0, 0, 1, 0) \in \mathfrak{e}_8^C.$$

Then since we have that $\phi_3(A)(0, 0, 0, 0, i, 0) = (0, 0, 0, 0, 1, 0)$, there exist no $A \in SO(2)$ such that $\varphi_{\text{E9}}(A, \varphi_{\text{Es}}(B))\rho_v = 1$. Hence, the right hand case is impossible. Thus we have that

$$\text{Ker } \varphi_{4215} = \{((E, E), 1), ((E, -E), 1), ((-E, -e_1E), 1), ((-E, e_1E), 1)\} \cong (\mathbf{Z}_4, 1).$$

Therefore we have the required isomorphism

$$(E_8)^{\lambda\omega\gamma} \cap (E_8)^{\lambda\omega\gamma\nu} \cong (SO(2) \times SU(8))/\mathbf{Z}_4 \rtimes \mathbf{Z}_2.$$

□

4.22. Type EVIII-IX-IX. In this section, we use a pair of involutive inner automorphisms $\tilde{\sigma}$ and $\tilde{\nu}$.

Lemma 4.22.1. (1) *The Lie algebra $(\mathfrak{e}_8)^\nu$ of the group $(E_8)^\nu$ is given by*

$$(\mathfrak{e}_8)^\nu = \{(\Phi, 0, 0, r, s, -\tau s) \mid \Phi \in \mathfrak{e}_7, r \in i\mathbf{R}, s \in \mathbf{C}\}.$$

(2) *The Lie algebra $(\mathfrak{e}_8)^{\nu\sigma}$ of the group $(E_8)^{\nu\sigma}$ is given by*

$$(\mathfrak{e}_8)^{\nu\sigma} = \{(\Phi, \tau\lambda Q, Q, r, s, -\tau s) \mid \Phi \in (\mathfrak{e}_7)^\sigma \cong \mathfrak{su}(2) \oplus \mathfrak{so}(12), Q \in (\mathfrak{P}^C)_{-\sigma}, r \in i\mathbf{R}, s \in \mathbf{C}\},$$

where $(\mathfrak{P}^C)_{-\sigma} = \{P \in \mathfrak{P}^C \mid \sigma P = -P\}$.

In particular, we have that

$$\dim((\mathfrak{e}_8)^\nu) = 133 + 1 + 2 = 136 = (3 + 66) + (8 + 8) \times 2 \times 2 + 1 + 2 = \dim((\mathfrak{e}_8)^{\nu\sigma}).$$

Proof. By straightforward computation, we can easily prove this lemma. □

From Lemma 4.22.1 and [13, Lemma 5.3.3], we have the following proposition.

Proposition 4.22.2. *The group $(E_8)^\nu$ is isomorphic to the group $(E_8)^{\nu\sigma}$: $(E_8)^\nu \cong (E_8)^{\nu\sigma}$.*

Remark. The author can not find any element $\delta \in E_8$ which gives the conjugation: $\nu\delta = \delta(\nu\sigma)$.

From the results of types EVII, EIX in Table 2 and Propositions 4.20.2, 4.22.2. we have the following theorem.

Theorem 4.22.3. For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \sigma\} \times \{1, \nu\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(E_8/(E_8)^\sigma, E_8/(E_8)^\nu, E_8/(E_8)^{\nu\sigma}) = (E_8/(E_8)^{\lambda_\omega}, E_8/(E_8)^\nu, E_8/(E_8)^\nu)$, that is, type (EVIII, EIX, EIX), abbreviated as EVIII-IX-IX.

Now, we determine the structure of the group $(E_8)^\sigma \cap (E_8)^\nu$.

Theorem 4.22.4. We have that $(E_8)^\sigma \cap (E_8)^\nu \cong (SU(2) \times SU(2) \times Spin(12))/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, $\mathbb{Z}_2 = \{(E, E, 1), (-E, E, -1)\}$, $\mathbb{Z}_2 = \{(E, E, 1), (E, -E, -\sigma)\}$.

Proof. We define a mapping $\varphi_{4224} : SU(2) \times SU(2) \times Spin(12) \rightarrow (E_8)^\sigma \cap (E_8)^\nu$ by

$$\varphi_{4224}(A, B, \beta) = \phi_3(A)\phi_2(B)\beta,$$

where ϕ_3, ϕ_2 are defined in Theorems 3.5.2, 3.4.2, respectively. From $\sigma\phi_3(A)\sigma = \phi_3(A)$ (Lemma 4.21.4 (3)) and $\phi_2(B)\beta \in (E_7)^\sigma$ (Theorem 3.4.2), it is easily to verify that φ_{4224} is well-defined. Since $\phi_3(A)$ commutes with $\phi_2(B)$ and β each other (see [8, Theorem 5.7.6] in detail), moreover $\phi_2(B)$ commutes with β in $E_7 \subset E_8$ (see [10, Theorem 4.11.15] in detail), we see that φ_{4224} is a homomorphism.

We shall show that φ_{4224} is surjection. Let $\alpha \in (E_8)^\sigma \cap (E_8)^\nu$. From $(E_8)^\sigma \cap (E_8)^\nu \subset (E_8)^\nu$, there exist $A \in SU(2)$ and $\delta \in E_7$ such that $\alpha = \varphi_{E_9}(A, \delta)$ (Theorem 3.5.2). Moreover, from $\alpha = \varphi_{E_9}(A, \delta) \in (E_8)^\sigma$, that is, $\sigma\varphi_{E_9}(A, \delta)\sigma = \varphi_{E_9}(A, \delta)$, again using $\sigma\phi_3(A)\sigma = \phi_3(A)$, we have that $\varphi_{E_9}(A, \sigma\delta\sigma) = \varphi_{E_9}(A, \delta)$. Hence, it follows that

$$\begin{cases} A = A \\ \sigma\delta\sigma = \delta \end{cases} \quad \text{or} \quad \begin{cases} A = -A \\ \sigma\delta\sigma = -\delta. \end{cases}$$

In the latter case, this case is impossible because of $A = 0$. In the former case, we see that $\delta \in (E_7)^\sigma \cong (SU(2) \times Spin(12))/\mathbb{Z}_2$. Hence, there exist $B \in SU(2)$ and $\beta \in Spin(12)$ such that $\delta = \varphi_{E_6}(B, \beta) = \phi_2(B)\beta$ (Theorem 3.4.2). Thus φ_{4224} is surjection.

Finally, we shall determine $\text{Ker } \varphi_{4224}$. From $\text{Ker } \varphi_{E_6} = \{(E, 1), (-E, -\sigma)\}$, we have that

$$\begin{aligned} \text{Ker } \varphi_{4224} &= \{(A, B, \beta) \in SU(2) \times SU(2) \times Spin(12) \mid \varphi_{4224}(A, B, \beta) = 1\} \\ &= \{(A, B, \beta) \in SU(2) \times SU(2) \times Spin(12) \mid \phi_3(A)\phi_2(B)\beta = 1\} \\ &= \{(A, B, \beta) \in SU(2) \times SU(2) \times Spin(12) \mid A = \pm E, \phi_2(B)\beta = \pm 1\} \\ &= \{(E, E, 1), (E, -E, -\sigma), (-E, E, -1), (-E, -E, \sigma)\} \\ &= \{(E, E, 1), (-E, E, -1)\} \times \{(E, E, 1), (E, -E, -\sigma)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2. \end{aligned}$$

Therefore we have the required isomorphism

$$(E_8)^\sigma \cap (E_8)^\nu \cong (SU(2) \times SU(2) \times Spin(12))/(\mathbb{Z}_2 \times \mathbb{Z}_2).$$

□

4.23. Type EIX-IX-IX. In this section, we use a pair of involutive inner automorphisms $\tilde{\nu}$ and $\tilde{\iota}_\omega$.

We define C -linear transformations $\iota_\omega, \nu\iota_\omega$ of \mathfrak{e}_8^C by

$$\begin{aligned} \iota_\omega(\Phi, P, Q, r, s, t) &= (\iota\Phi\iota^{-1}, \iota Q, -\iota P, -r, -t, -s), \\ \nu\iota_\omega(\Phi, P, Q, r, s, t) &= (\iota\Phi\iota^{-1}, -\iota Q, \iota P, -r, -t, -s), \quad (\Phi, P, Q, r, s, t) \in \mathfrak{e}_8^C, \end{aligned}$$

where ι of the right hand side is same one as $\iota \in E_7$. Then we see that $\iota_\omega, \nu\iota_\omega \in E_8, \iota_\omega^2 = \nu\iota_\omega^2 = 1$. Hence $\iota_\omega, \nu\iota_\omega$ induce involutive inner automorphisms $\tilde{\iota}_\omega, \tilde{\nu}\tilde{\iota}_\omega$ of E_8 : $\tilde{\iota}_\omega(\alpha) = \iota_\omega\alpha\iota_\omega, \tilde{\nu}\tilde{\iota}_\omega(\alpha) = (\nu\iota_\omega)\alpha(\iota_\omega\nu), \alpha \in E_8$.

Lemma 4.23.1. (1) The Lie algebra $(\mathfrak{e}_8)^{\iota_\omega}$ of the group $(E_8)^{\iota_\omega}$ is given by

$$(\mathfrak{e}_8)^{\iota_\omega} = \left\{ (\Phi, \tau\lambda Q, Q, 0, s, -s) \mid \begin{array}{l} \Phi \in (\mathfrak{e}_7)^\iota \cong \mathfrak{u}(1) \oplus \mathfrak{e}_6, Q = (X, i\tau X, \xi, i\tau\xi), \\ X \in \mathfrak{J}^C, \xi \in C, s \in \mathbf{R} \end{array} \right\}.$$

(2) The Lie algebra $(\mathfrak{e}_8)^{\nu_\omega}$ of the group $(E_8)^{\nu_\omega}$ is given by

$$(\mathfrak{e}_8)^{\nu_\omega} = \left\{ (\Phi, \tau\lambda Q, Q, 0, s, -s) \mid \begin{array}{l} \Phi \in (\mathfrak{e}_7)^\iota \cong \mathfrak{u}(1) \oplus \mathfrak{e}_6, Q = (X, -i\tau X, \xi, -i\tau\xi), \\ X \in \mathfrak{J}^C, \xi \in C, s \in \mathbf{R} \end{array} \right\}.$$

In particular,

$$\begin{aligned} \dim((\mathfrak{e}_8)^{\iota_\omega}) &= (1 + 78) + (27 + 1) \times 2 + 1 = 136 \\ &= (1 + 78) + (27 + 1) \times 2 + 1 = \dim((\mathfrak{e}_8)^{\nu_\omega}). \end{aligned}$$

Proof. By straightforward computation, we can easily prove this lemma. \square

From Lemmas 4.22.1 (1), 4.23.1 above and [13, Lemma 5.3.3], we have the following proposition.

Proposition 4.23.2. The group $(E_8)^\nu$ is isomorphic to both of the groups $(E_8)^{\iota_\omega}$ and $(E_8)^{\nu_\omega}$: $(E_8)^\nu \cong (E_8)^{\iota_\omega} \cong (E_8)^{\nu_\omega}$.

Remark. The author can not find any element $\delta, \delta' \in E_8$ which give the conjugations: $\nu\delta = \delta\iota_\omega$, $\iota_\omega\delta' = \delta'\nu_\omega$.

From the result of type EIX in Table 2 and Proposition 4.23.2, we have the following theorem.

Theorem 4.23.3. For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \nu\} \times \{1, \iota_\omega\}$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space is of type $(E_8/(E_8)^\nu, E_8/(E_8)^{\iota_\omega}, E_8/(E_8)^{\nu_\omega}) = (E_8/(E_8)^\nu, E_8/(E_8)^\nu, E_8/(E_8)^\nu)$, that is, type (EIX, EIX, EIX), abbreviated as EIX-IX-IX.

Consider a group $\mathcal{Z}_2 = \{1, \nu\}$, where $\nu = \delta_\nu \lambda$ (δ_ν and λ are defined in Section 4.21 and Section 3.4, respectively). Then the group \mathcal{Z}_2 acts on the group $SO(2) \times U(1) \times E_6$ by

$$\nu(A, \theta, \beta) = ((iI)A(iI)^{-1}, \theta^{-1}, \tau\beta\tau),$$

and let $(SO(2) \times U(1) \times E_6) \rtimes \mathcal{Z}_2$ be the semi-direct product $SO(2) \times U(1) \times E_6$ and \mathcal{Z}_2 with this action.

Now, we determine the structure of the group $(E_8)^\nu \cap (E_8)^{\iota_\omega}$.

Theorem 4.23.4. We have that $(E_8)^\nu \cap (E_8)^{\iota_\omega} \cong (SO(2) \times U(1) \times E_6) / (\mathcal{Z}_2 \times \mathcal{Z}_3) \rtimes \mathcal{Z}_2$, $\mathcal{Z}_2 = \{(E, 1, 1), (-E - 1, 1)\}$, $\mathcal{Z}_3 = \{(E, 1, 1), (E, \omega, \phi_2(\omega^2)), (E, \omega^2, \phi_2(\omega))\}$, $\mathcal{Z}_2 = \{1, \nu\}$.

Proof. We define a mapping $\varphi_{4234} : (SO(2) \times U(1) \times E_6) \rtimes \mathbb{Z}_2 \rightarrow (E_8)^\nu \cap (E_8)^{\iota_\omega}$ by

$$\begin{aligned} \varphi_{4234}((A, \theta, \beta), 1) &= \varphi_{E_9}(A, \varphi_{E_7}(\theta, \beta)), \\ \varphi_{4234}((A, \theta, \beta), \nu) &= \varphi_{E_9}(A, \varphi_{E_7}(\theta, \beta)) \nu, \end{aligned}$$

where φ_{E_7} are defined in Theorem 3.4.3. From $\varphi_{E_7}(\theta, \beta) \in (E_7)^\iota$ and $\nu\nu = \nu\nu$, it is clear that $\varphi_{4234}((A, \theta, \beta), 1), \varphi_{4234}((A, \theta, \beta), \nu) \in (E_8)^\nu$, moreover from $\varphi_{E_9}(A, \varphi_{E_7}(\theta, \beta)) = \phi_3(A)\phi(\theta)\beta$ and $\iota_\omega\nu = \nu\iota_\omega$, it is easily to verify that $\varphi_{4234}((A, \theta, \beta), 1), \varphi_{4234}((A, \theta, \beta), \nu) \in (E_8)^{\iota_\omega}$. Hence φ_{4234} is well-defined. By straightforward computation, we can confirm that φ_{4234} is a homomorphism. Indeed, we show that the case of $\varphi_{4234}((A, \theta, \beta), \nu) \varphi_{4234}((B, \zeta, \kappa), 1) = \varphi_{4234}(((A, \theta, \beta), \nu) ((B, \zeta, \kappa), 1))$ as example. For the left hand side of this equality, we have that

$$\begin{aligned} \varphi_{4234}((A, \theta, \beta), \nu) \varphi_{4234}((B, \zeta, \kappa), 1) &= \varphi_{E_9}(A, \varphi_{E_7}(\theta, \beta)) \nu \varphi_{E_9}(B, \varphi_{E_7}(\zeta, \kappa)) \\ &= \phi_3(A)\varphi_{E_7}(\theta, \beta) \nu \phi_3(B)\varphi_{E_7}(\zeta, \kappa). \end{aligned}$$

On the other hand, for the right hand side of same one, using $\delta_v = \phi_3(iI)$, $\delta_v \lambda = \lambda \delta_v$ (Lemmas 4.21.4 (1), 4.21.1) and $\tau \kappa \tau = \lambda \kappa \lambda^{-1}$, that is, $(\tau \kappa \tau) \lambda = \lambda \kappa$ as $\kappa \in E_6 \subset E_7$, we have that

$$\begin{aligned}
 \varphi_{4234}(((A, \theta, \beta), \nu)((B, \zeta, \kappa), 1)) &= \varphi_{4234}(((A, \theta, \beta)(\nu(B, \zeta, \kappa))), \nu) \\
 &= \varphi_{4234}(((A, \theta, \beta)((iI)B(iI)^{-1}, \zeta^{-1}, \tau \kappa \tau)), \nu) \\
 &= \varphi_{4234}((A(iI)B(iI)^{-1}, \theta \zeta^{-1}, \beta \tau \kappa \tau), \nu) \\
 &= \varphi_{E_9}((A(iI)B(iI)^{-1}, \varphi_{E_7}(\theta \zeta^{-1}, \beta \tau \kappa \tau), \nu) \\
 &= \phi_3(A(iI)B(iI)^{-1}) \varphi_{E_7}(\theta \zeta^{-1}, \beta \tau \kappa \tau) \nu \\
 &= \phi_3(A(iI)B(iI)^{-1}) \varphi_{E_7}(\theta \zeta^{-1}, \beta \tau \kappa \tau)(\delta_v \lambda) \\
 &= \phi_3(A)(\delta_v \phi_3(B) \delta_v^{-1}) (\phi(\theta) \phi(\zeta^{-1}) (\beta \tau \kappa \tau) (\delta_v \lambda)) \\
 &= \phi_3(A)(\delta_v \phi_3(B) \delta_v^{-1}) \phi(\theta) \beta \phi(\zeta^{-1}) \tau \kappa \tau (\lambda \delta_v) \\
 &= \phi_3(A)(\delta_v \phi_3(B) \delta_v^{-1}) \phi(\theta) \beta (\lambda \phi(\zeta) \lambda^{-1}) \lambda \kappa \delta_v \\
 &= \phi_3(A)(\delta_v \phi_3(B) \delta_v^{-1}) \phi(\theta) \beta \lambda \phi(\zeta) \kappa \delta_v \\
 &= \phi_3(A)(\delta_v \phi_3(B) \delta_v^{-1}) \phi(\theta) \beta (\lambda \delta_v) \phi(\zeta) \kappa \\
 &= \phi_3(A) \phi(\theta) \beta (\delta_v \phi_3(B) \delta_v^{-1}) (\delta_v \lambda) \phi(\zeta) \kappa \\
 &= \phi_3(A) \phi(\theta) \beta \delta_v \phi_3(B) \lambda \phi(\zeta) \kappa \\
 &= \phi_3(A) \phi(\theta) \beta (\delta_v \lambda) \phi_3(B) \phi(\zeta) \kappa \\
 &= \phi_3(A) \varphi_{E_7}(\theta, \beta) \nu \phi_3(B) \varphi_{E_7}(\zeta, \kappa),
 \end{aligned}$$

where ϕ is defined in Theorem 3.4.3. Similarly, the other cases are shown.

We shall show that φ_{4234} is surjection. Let $\alpha \in (E_8)^\nu \cap (E_8)^{\iota\omega}$. From $(E_8)^\nu \cap (E_8)^{\iota\omega} \subset (E_8)^\nu$, there exist $A \in SU(2)$ and $\delta \in E_7$ such that $\alpha = \varphi_{E_9}(A, \delta)$ (Theorem 3.5.2). Moreover, since $\alpha = \varphi_{E_9}(A, \delta) \in (E_8)^{\iota\omega}$, that is, $\iota_\omega \varphi_{E_9}(A, \delta) \iota_\omega^{-1} = \varphi_{E_9}(A, \delta)$, using $\iota_\omega \phi_3(A) \iota_\omega^{-1} = \phi_3({}^t A^{-1})$ (Lemma 4.21.4 (2)), we have that $\varphi_{E_9}({}^t A^{-1}, \iota \delta \iota^{-1}) = \varphi_{E_9}(A, \delta)$. Hence, it follows that

$$\begin{cases} {}^t A^{-1} = A \\ \iota \delta \gamma \iota^{-1} = \delta \end{cases} \quad \text{or} \quad \begin{cases} {}^t A^{-1} = -A \\ \iota \delta \gamma \iota^{-1} = -\delta. \end{cases}$$

In the former case, we see that $A \in SO(2)$ and $\delta \in (E_7)^t \cong (U(1) \times E_6)/\mathbb{Z}_3$. Hence, there exist $\theta \in U(1)$ and $\beta \in E_6$ such that $\delta = \varphi_{E_7}(\theta, \beta)$ (Theorem 3.4.3). Thus we have that $\alpha = \varphi_{E_9}(A, \delta) = \varphi_{E_9}(A, \varphi_{E_7}(\theta, \beta)) = \varphi_{4234}((A, \theta, \beta), 1)$. In the latter case, we see that $A = A'(iI)$, $A' \in SO(2)$ and $\delta = \delta' \lambda$, $\delta' \in (E_7)^t$. Hence, as in the former case, we have that

$$\begin{aligned}
 \alpha &= \varphi_{E_9}(A, \delta) = \varphi_{E_9}(A(iI), \delta' \lambda) = \phi_3((A(iI))(\delta' \lambda)) = \phi_3(A') \phi_3(iI)(\delta' \lambda) \\
 &= \phi_3(A') \delta' (\phi_3(iI) \lambda) = \phi_3(A') \delta' (\delta_v \lambda) = \varphi_{E_9}(A', \delta') (\delta_v \lambda) \\
 &= \varphi_{E_9}(A', \varphi_{E_7}(\theta', \beta')) \nu = \varphi_{4234}((A', \theta', \beta'), \nu).
 \end{aligned}$$

Thus φ_{4234} is surjection.

Finally, we shall determine $\text{Ker } \varphi_{4234}$. From the definition of kernel, it is as follows:

$$\begin{aligned}
 \text{Ker } \varphi_{4234} &= \{((A, \theta, \beta), 1) \mid \varphi_{4234}((A, \theta, \beta), 1) = 1\} \cup \{((A, \theta, \beta), \nu) \mid \varphi_{4234}((A, \theta, \beta), \nu) = 1\} \\
 &= \{((A, \theta, \beta), 1) \mid \varphi_{E_9}(A, \varphi_{E_7}(\theta, \beta)) = 1\} \cup \{((A, \theta, \beta), \nu) \mid \varphi_{E_9}(A, \varphi_{E_7}(\theta, \beta)) \nu = 1\}.
 \end{aligned}$$

Here, for the left hand side case, we have that

$$\begin{aligned}
& \{((A, \theta, \beta), 1) \mid \varphi_{E_9}(A, \varphi_{E_7}(\theta, \beta)) = 1\} \\
&= \{((A, \theta, \beta), 1) \mid A = \pm E, \varphi_{E_7}(\theta, \beta) = \pm 1\} \\
&= \{((A, \theta, \beta), 1) \mid A = \pm E, \phi(\theta)\beta = \pm 1\} \\
&= \{((E, 1, 1), 1), (E, \omega, \phi_2(\omega^2)), (E, \omega^2, \phi_2(\omega), \\
&\quad ((-E - 1, 1), 1), ((-E, -\omega, \phi_2(\omega^2)), 1), ((-E, -\omega^2, \phi_2(\omega), 1)\} \\
&= \{(E, 1, 1), (-E - 1, 1)\} \times \{(E, 1, 1), (E, \omega, \phi_2(\omega^2)), (E, \omega^2, \phi_2(\omega))\}.
\end{aligned}$$

For the right hand case, in a similar way as the argument of kernel in Theorem 4.21.5, we have that $\{((A, \theta, \beta), \nu) \mid \varphi_{E_9}(A, \varphi_{E_7}(\theta, \beta))\nu = 1\} = \phi$. Thus we can obtain that

$$\text{Ker } \varphi_{4234} = \{(E, 1, 1), (-E - 1, 1)\} \times \{(E, 1, 1), (E, \omega, \phi_2(\omega^2)), (E, \omega^2, \phi_2(\omega))\} \cong \mathbf{Z}_2 \times \mathbf{Z}_3.$$

Therefore we have the required isomorphism

$$(E_8)^v \cap (E_8)^{\omega} \cong (SO(2) \times U(1) \times E_6)/(\mathbf{Z}_2 \times \mathbf{Z}_3) \rtimes \mathbf{Z}_2.$$

□

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